Autoresonance model of teraherz waves generator

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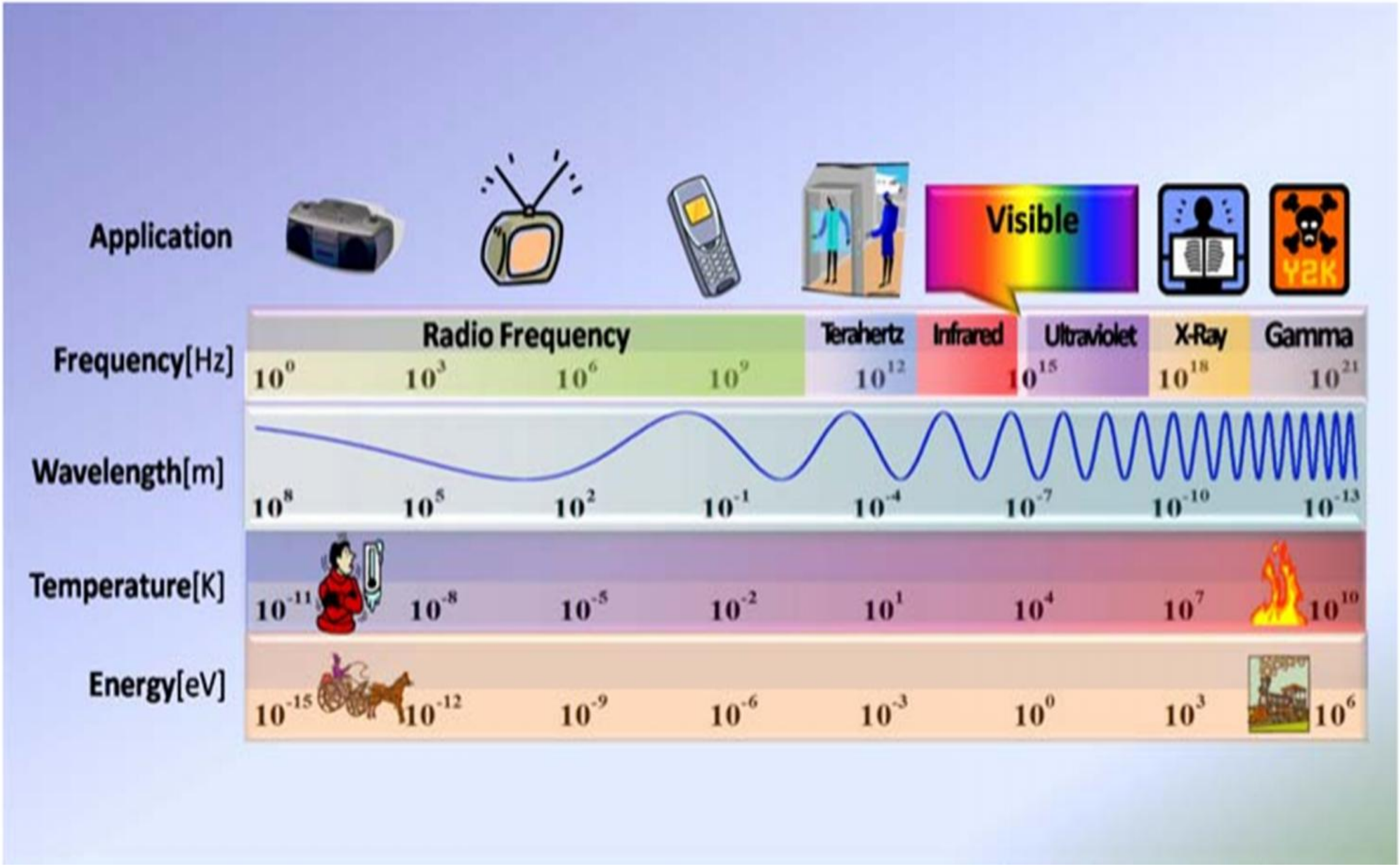
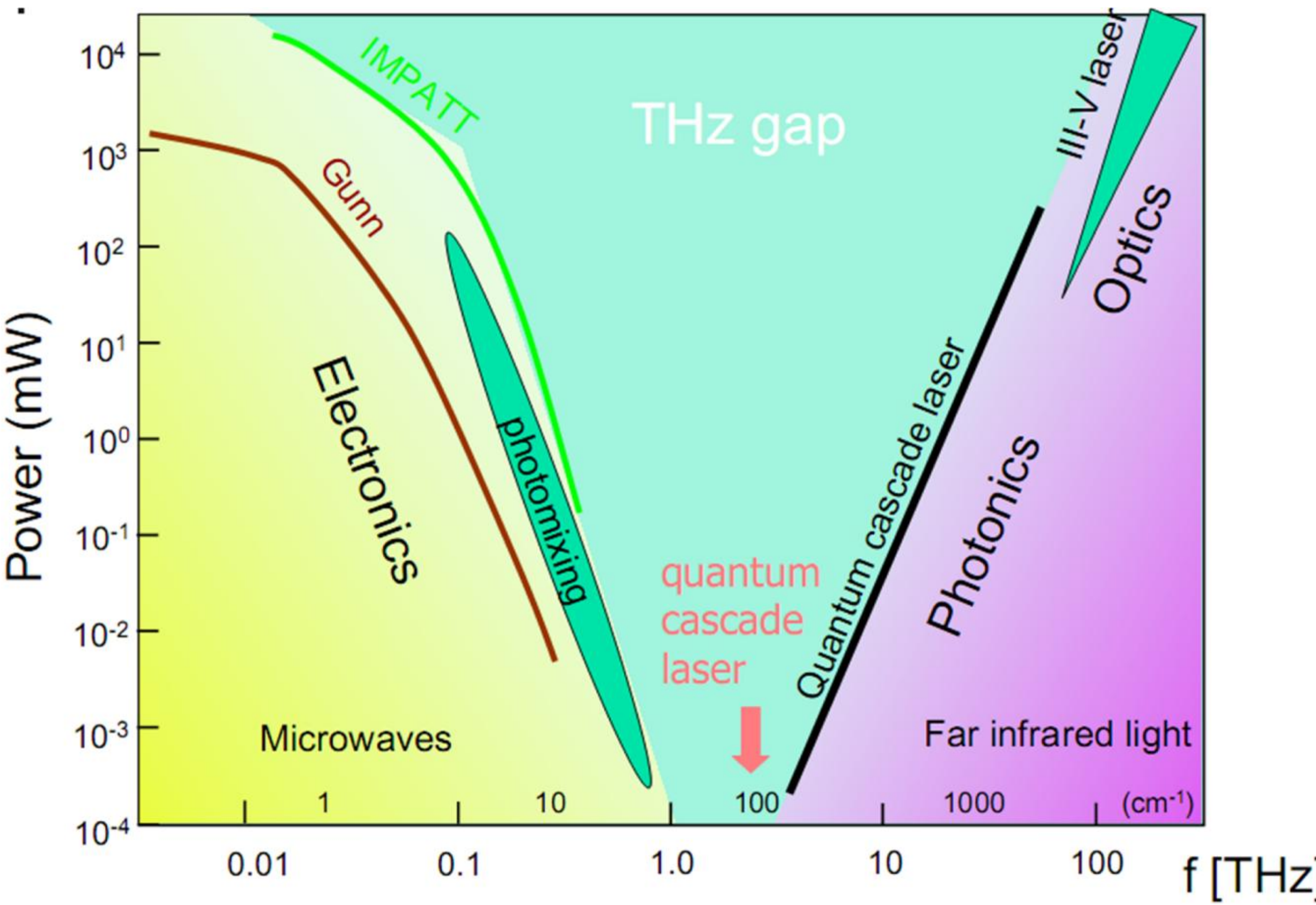
Electromagnetic spectrum



Spectrum coverage by up-to-date devices



S.Z.Lin and X.Hu, Materials Nanoarchitectonics Int. Conf. (2008)

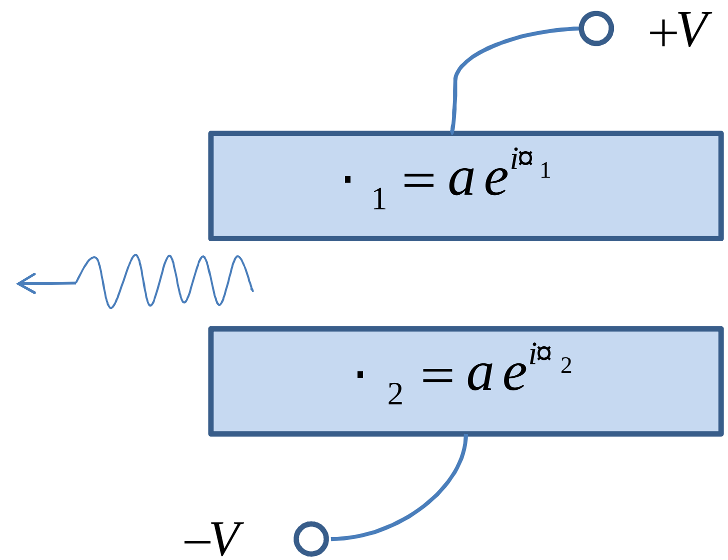
Josephson effect

Schrödinger equations on $\cdot_1, \cdot_2$

$$\begin{cases} \hbar \dfrac{d¤_1}{dt} = c_1 \sin\left(¤_1 - ¤_2\right) + eV, \\ \hbar \dfrac{d¤_2}{dt} = c_1 \sin\left(¤_1 - ¤_2\right) - eV \end{cases}$$

where $¤_1, ¤_2$ - phases of electric field

Josephson junction (1962)



Josephson current $\quad J = J_0 V \dfrac{e}{\hbar} \sin µ\, t$

Radiation frequency $\quad \dfrac{d}{dt}\left(¤_1 - ¤_2\right) = µ = V \dfrac{2e}{\hbar}$

$$\frac{2e}{\hbar} = 0.483 \frac{[THz]}{[mV]} \quad \Rightarrow \quad µ \approx 0.5 THz, \quad V \approx 1 mV$$

The radiation power in single junction is very small $\sim 10^{-6}$ milliwatt !

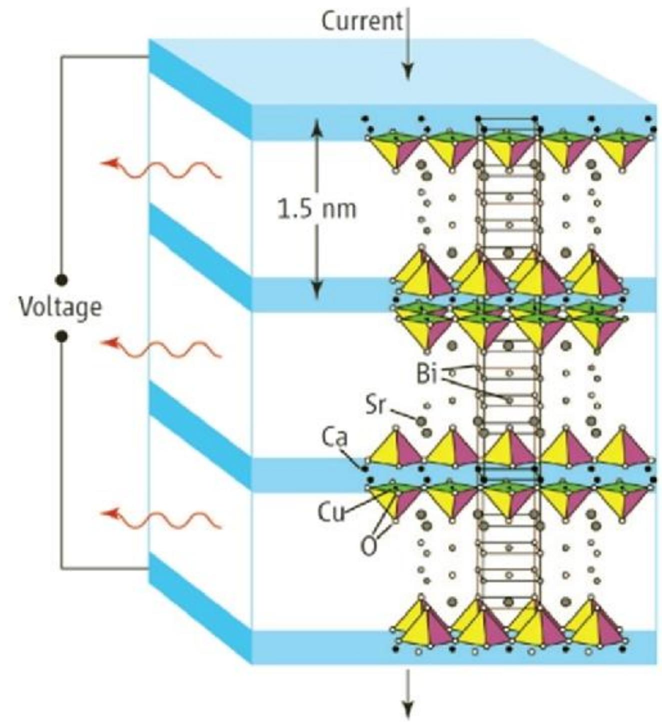
Coupled Josephson stack

Layered high-temperature superconductors (1986)

- Superconducting layer CuO_2
- Insulating layers BiO and SrO

⇩

Natural multi-layered stack of Josephson junctions



High T_c material $Bi_2Sr_2CaCu_2O_8$

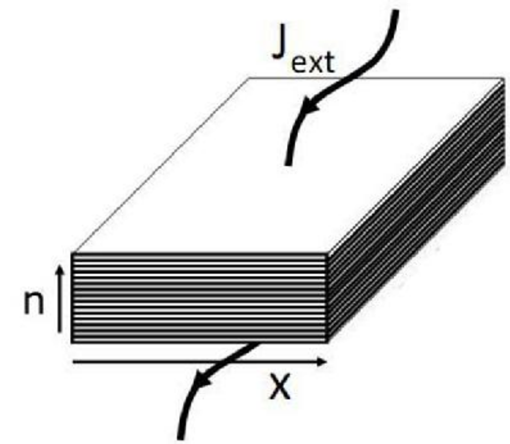
Experiment:

• High critical temperature	$T_c \approx 104$ K
• Number of layers	$N \approx 200\,000$
• Emitted frequency	$\omega_{res} \approx 0.785$ Thz
• Emission power	$P \approx 0.5$ mW

L.Ozyuzer et al., Science 318, 1291 (2007)

Mathematical model of Josephson stack

Coupled sine-Gordon equations



$$l^2 \frac{\partial^2 ¤_n}{\partial x^2} = (1 - \sim\Delta_2)\left(\frac{\partial^2 ¤_n}{\partial t^2} + \sin ¤_n + \quad \frac{\partial ¤_n}{\partial t} - j_{ext}\right),$$

$$\frac{\partial ¤_n(t,-L)}{\partial x} = \frac{\partial ¤_n(t,L)}{\partial x} = 0, \quad -L < x < L, \quad n = 1, 2, \ldots, N,$$

where $¤_n$ is phase of electric field in the *n*-th layer,

$µ_{res} = ® l / L$ - resonant frequency, j_{ext} - external voltage,

$\Delta_2 ¤_n = ¤_{n+1} - 2¤_n + ¤_{n-1}$ - discrete Laplace operator,

$\ll 1$ - small dissipation parameter, $l, \sim = const, \quad l \gg 1$

Find asymptotic solution as $N \gg 1$ with autoresonant behavior

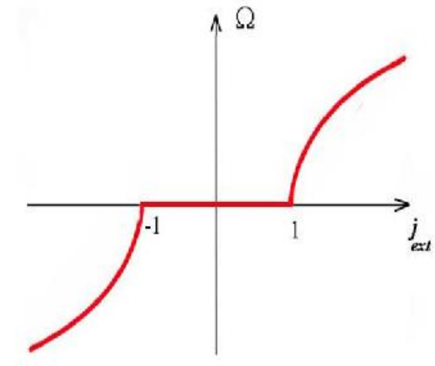
$$¤_n \to O(µt) \quad \text{as} \quad t \to \infty$$

X. Hu and S. Lin, Phys. Rev. B, V.78 N.13, 134510 (2008)
L.Bulaevskii, A.Koshelev, Phys.Rev.Lett.,V. 99, 057002 (2007)

Model autoresonant equation

Phase equation for single Josephson junction

$$\frac{d¤}{dt} = \sim\left(-\sin ¤ + j_{ext}\right), \qquad \sim, j_{ext} = \text{const}$$



period-tuning , or voltage-current graph

Period $\quad T = \int_0^{2®} \frac{d¤}{\sim\left(-\sin ¤ + j_{ext}\right)}$ $\qquad$ Frequency $\quad \Omega = \frac{2®}{T}$

Let us simulate neighboring junctions by the periodic force

$$\frac{d¤}{dt} = \sim\left(-\sin ¤ + j_{ext} + A\cos(µt)\right), \quad A \gg 1$$

Resonance condition $\quad k\Omega + lµ = 0 \quad \Rightarrow \quad$ Linear growth of ¤ in time

Calculate it in terms of coefficients $\quad \sim, j_{ext}, A$

$$¤(t) = nµt + A\cdot_{-1}(t) + \cdot_{0}(t) + O(A^{-1}),$$

$\langle \dot{\cdot}_{-1} \rangle = \langle \dot{\cdot}_{0} \rangle = 0$ - asymptotic ansatz

Model autoresonant equation (continued)

Scaling in leading orders :

$$A^1: \quad \cdot_{-1} = \frac{\sim}{\mu}\sin(n\mu t) + C,$$

$$A^0: \quad \frac{d\cdot_0}{dt} = \sim j_{ext} - n\mu - \sim \sin\left(n\mu t + \frac{\sim A}{\mu}\sin(\mu t) + AC\right)$$

Integrate over period $T = \frac{2®}{\mu}$ and use expansion

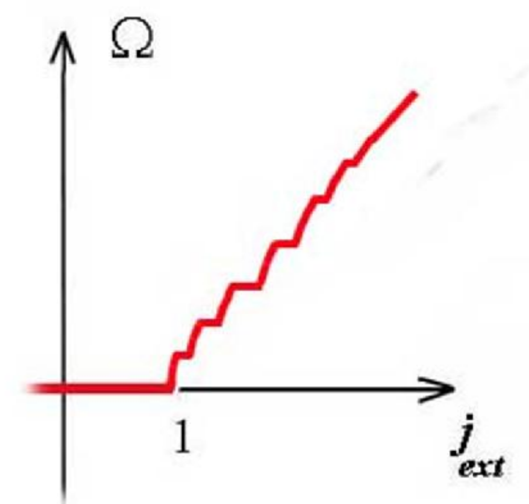
$$\exp[i(n\mu t + x\sin\mu t + y)] = \sum_{k=-\infty}^{\infty} J_k(x)\exp[i((k+n)\mu t + y)]$$

$$\langle \dot{\cdot}_0 \rangle = 0 \quad \Rightarrow \quad \boxed{0 = \sim j_{ext} - n\mu - \sim \sin(AC) J_{-n}\left(\frac{\sim A}{\mu}\right),}$$

$J_{-n}(x) = (-1)^n J_n(x)$ - Bessel function, $|\sin(AC)| < 1$, then

Autoresonant condition: $\left| j_{ext} - n\frac{\mu}{\sim} \right| < \left| J_n\left(\frac{\sim A}{\mu}\right) \right|$

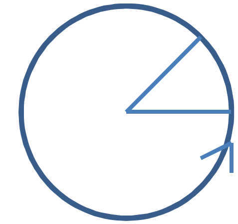
“Shapiro steps”



S. Shapiro, *Phys. Rev. Lett.* **11**, 80 (1963)

Iterations of circle maps

$$¤_{n+1} = ¤_n + £ \sin ¤_n + j, \quad £, j = const$$



$$¤_n = 2® n° + ¤_0 + O(n^{-1}), \quad °$$ - rotation number

$$£ = 0 \quad \Rightarrow \quad ° = j / 2®$$

$°$ - irrational $\Rightarrow$ $¤_n$ is quasi-periodic, unstable rotation

$° = p / q \Rightarrow ¤_{n+q} = ¤_n + 2® p$ - stable periodic rotation

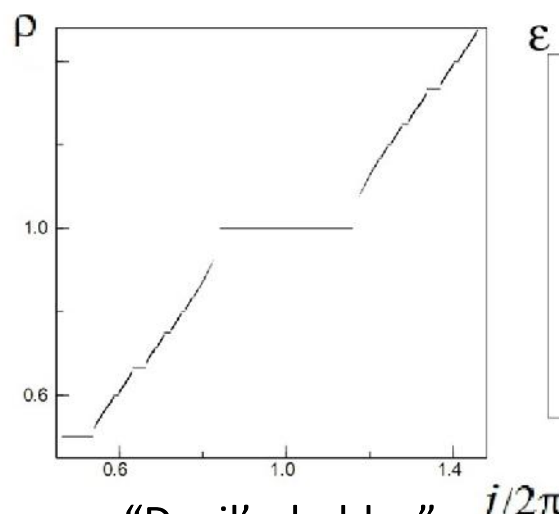
Denjoy theorem (1932)

If $°$ is irrational, then there exists a map

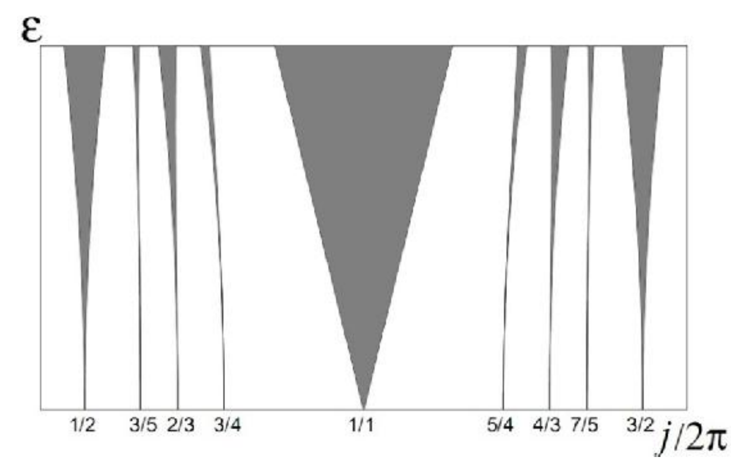
$$¤ = g(¯),$$

such that $g(¯ + 2®) = g(¯)$

and $¯_{n+1} = ¯_n + 2®°$



"Devil's ladder"



"Arnold tongues"

A. Pikovsky, M. Rosenblum, J. Kurths, Synchronization, Cambridge, 2003

Formal asymptotics

$$l^2 \frac{\partial^2 ¤_n}{\partial x^2} = (1 - \sim\Delta_2)\left(\frac{\partial^2 ¤_n}{\partial t^2} + \sin ¤_n + \quad \frac{\partial ¤_n}{\partial t} - j_{ext}\right),$$

$$\frac{\partial ¤_n(t,-L)}{\partial x} = \frac{\partial ¤_n(t,L)}{\partial x} = 0, \quad -L < x < L, \quad n = 1, 2, \dots, N,$$

Asymptotic ansatz

$$¤_n(x,t) = \mu t + A\cos\frac{® x}{L}\sin(\mu t + ¨) + (-1)^n u(x) + \\ + \sum_{m=2}^{\infty} A_m \cos\frac{® xm}{L}\sin(\mu t + ¨_m) + O(\mu^{-2}).$$

- “Sine of sine” Bessel expansion

$$\exp\left[i(\mu t + (-1)^n u(x) + z\sin(\mu t + ¨))\right] = \sum_{k=-\infty}^{\infty} J_k(z)\exp\left[i\left((k+1)\mu t + k¨ + (-1)^n u(x)\right)\right].$$

- Action of Δ_2 operator

$$\Delta_2 f(x,t) = 0,$$

$$\Delta_2 \sin\left[f(x,t) + (-1)^n u(x)\right] = (-1)^n 4 \sin u(x) \cos f(x,t).$$

Leading order terms

(1) $$\frac{d^2u}{dx^2} = 4\sim J_1\left(A\cos\neg x\right)\cos\text{¨}\ \sin u,$$ where $\neg = \frac{\text{®}}{L}$

(2) $$j_{ext} = \mu + \frac{\sin\text{¨}}{2L}\int_{-L}^{L} J_1\left(A\cos\neg x\right)\cos u(x)dx,$$

(3) $$\mu A = \frac{\sin\text{¨}}{L}\int_{-L}^{L}\left[J_2\left(A\cos\neg x\right) + J_0\left(A\cos\neg x\right)\right]\cos\neg x\cos u(x)dx,$$

(4) $$A\left(\mu^2 - \ell^2\neg^2\right) = \frac{\cos\text{¨}}{L}\int_{-L}^{L}\left[-J_2\left(A\cos\neg x\right) + J_0\left(A\cos\neg x\right)\right]\cos\neg x\cos u(x)dx.$$

Since $zJ_{n-1}(z) + zJ_{n+1}(z) = 2nJ_n(z)$ integral (3) is reduced to integral (2)

$$j_{ext} = \mu\left(1 + \frac{A^2}{4}\right).$$

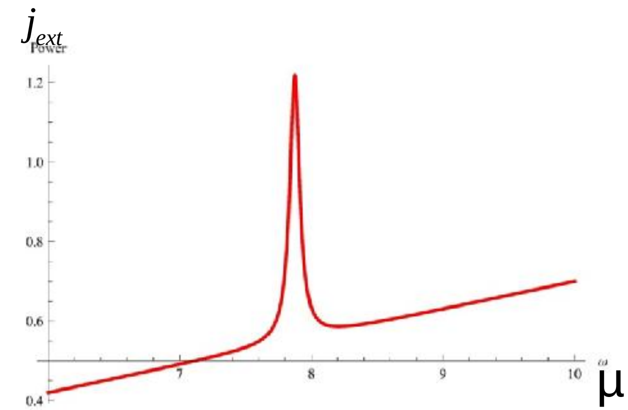
Autoresonant conditions

Express $j_{ext} = \mu\left(1+\frac{A^2}{4}\right)$ as current-frequency condition

Solve equation (4) to find A in terms of μ

For $A \ll 1$ Bessel functions J_0, J_2 are

$$J_0(A\cos¶) = 1 - \frac{A^2\cos^2¶}{4} + \ldots, \quad J_2(A\cos¶) = \frac{A^2\cos^2¶}{8} + \ldots$$



Resonant frequency

Then equations (2) – (4) yield

$$A^2 = \frac{4I_1^2}{\left(\mu^2 - ®^2\ell^2/L^2\right)^2 + {}^2\mu^2},$$

$$j_{ext} = \mu\left[1 + \frac{I_1^2}{\left(\mu^2 - ®^2\ell^2/L^2\right)^2 + {}^2\mu^2}\right],$$

$$\sin ¨ = \frac{\mu}{I_2},$$

where

$$I_1 = \frac{1}{2L}\int_{-L}^{L} \cos\frac{®x}{L}\cos u(x)dx,$$

$$I_2 = \frac{1}{2L}\int_{-L}^{L} \cos^2\frac{®x}{L}\cos u(x)dx.$$

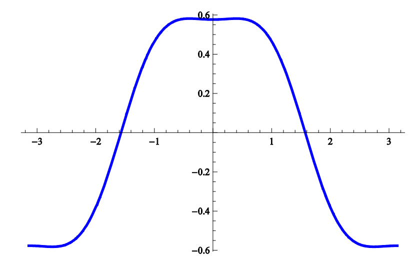
Main resonance equation

"Kapitsa pendulum" (1951)

$$£^{2}\frac{d^{2}u}{d¶^{2}} = J_{1}\left(A\cos ¶\right)\sin u, \quad -® < ¶ < ®$$

$$u'(-®) = u'(®) = 0$$

$$y = J_{1}\left(A\cos ¶\right)$$



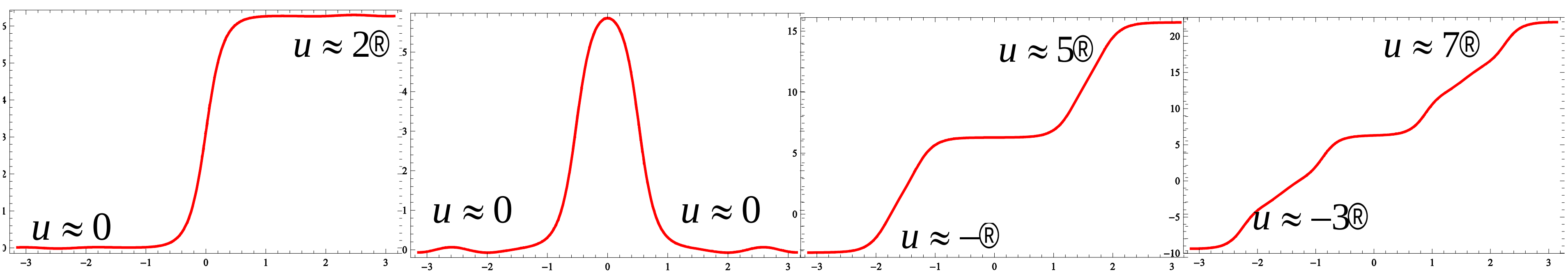
Full energy $E = \frac{1}{2}£^{2}\left(\frac{du}{d¶}\right)^{2} + J_{1}\left(A\cos ¶\right)\cos u$

$$¶ = \frac{®}{L}x, \quad £ = \frac{®}{2L\sqrt{~\cos ¨}}$$

Average energy $\left\langle E \right\rangle = \frac{1}{2®}\int_{-®}^{®} E(¶)\,d¶$

$$A = 2, \quad £ = 0.1334$$

$\left\langle E \right\rangle = 0.064731$ $\left\langle E \right\rangle = 0.124772$ $\left\langle E \right\rangle = 0.183888$ $\left\langle E \right\rangle = 0.328347$



Single kink has minimum energy!

Small perturpations of kinks

$$¤_{+}=\frac{1}{2}(¤_{2m+1}+¤_{2m}),\quad ¤_{-}=\frac{1}{2}(¤_{2m+1}-¤_{2m}),$$

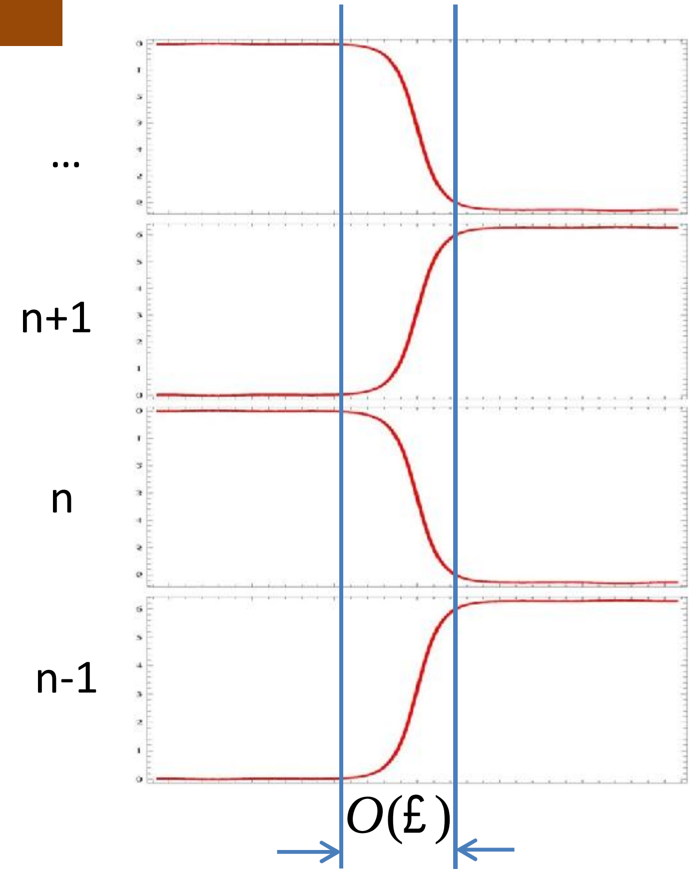
$$\begin{cases}\dfrac{\partial^2 ¤_+}{\partial t^2}-\ell^2\dfrac{\partial^2 ¤_+}{\partial x^2}+\quad\dfrac{\partial ¤_+}{\partial t}=-\sin ¤_+\cos ¤_- - j_{ext},\\ \dfrac{\partial^2 ¤_-}{\partial t^2}-\ell^2\dfrac{\partial^2 ¤_-}{\partial x^2}+\quad\dfrac{\partial ¤_-}{\partial t}=-\sin ¤_-\cos ¤_+ .\end{cases}$$

$$\frac{\partial ¤_{\pm}(-L,t)}{\partial x}=\frac{\partial ¤_{\pm}(L,t)}{\partial x}=0,\quad -L<x<L,$$

$$¤_{+}^{asym}(x,t)=µ\,t+A\cos\frac{®\,x}{L}\sin(µ t+¨)+o(µ^{-2}),$$

$$¤_{-}^{asym}(x,t)=(-1)^n u(x)+o(µ^{-2}).$$

Boundary conditions are satisfied!



Electric field $(-1)^n u(x)$ profile in subsequent layers

$$\cos ¤_{+}^{asym}=\cos(µ t+O(1)),$$

$$\cos ¤_{-}^{asym}=\cos u(x)+O(µ^{-2})=1+O(£^{2})+O(µ^{-2})$$

Small perturpations of kinks

$$\begin{cases} \dfrac{\partial^2 ¤_+}{\partial t^2} - \ell^2 \dfrac{\partial^2 ¤_+}{\partial x^2} + \dfrac{\partial ¤_+}{\partial t} = -\sin ¤_+ \cos ¤_- - j_{ext}, \\ \dfrac{\partial^2 ¤_-}{\partial t^2} - \ell^2 \dfrac{\partial^2 ¤_-}{\partial x^2} + \dfrac{\partial ¤_-}{\partial t} = -\sin ¤_- \cos ¤_+ . \end{cases}$$

$$\cos ¤_-^{asym} = 1 + O(£^2) + O(µ^{-2})$$

$$\frac{\partial^2 ¤_+^{asym}}{\partial t^2} + \frac{\partial ¤_+^{asym}}{\partial t} = -\sin ¤_+^{asym} - j_{ext}$$

$$¤_+ = ¤_+^{asym} + ¡_+$$

$$¤_- = ¤_-^{asym} + ¡_-$$

$$\begin{cases} \dfrac{\partial^2 ¡_+}{\partial t^2} - \ell^2 \dfrac{\partial^2 ¡_+}{\partial x^2} + \dfrac{\partial ¡_+}{\partial t} = -\sin ¡_+ \left(1 + £^2 f(x) + µ^{-2} g(x,t)\right), \\ \dfrac{\partial ¡_+(-L,t)}{\partial x} = \dfrac{\partial ¡_+(L,t)}{\partial x} = 0, \quad -L < x < L, \\ ¡_+(x,0) = \partial_t ¡_+(x,0) = 0. \end{cases}$$

Then $¡_+(x,t) = O\left(£^2\right) + O\left(µ^{-2}\right)$ and similarly

$$¡_-(x,t) = O\left(£^2\right) + O\left(µ^{-2}\right)$$

Open questions

- Strict proof of autoresonant asymptotics (initial discrete sine-Gordon chain)
- Prove a transition to continuous model (two coupled sine-Gordon equations)
- Analysis of solutions to main resonance equation

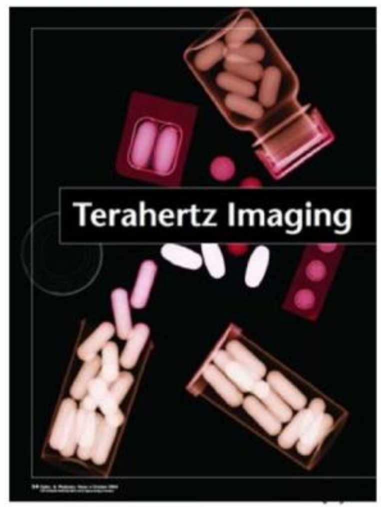
$$\varepsilon^2 \frac{d^2 u}{d\theta^2} = J_1\left(A\cos\theta\right)\sin u, \quad -\infty < \theta < \infty$$

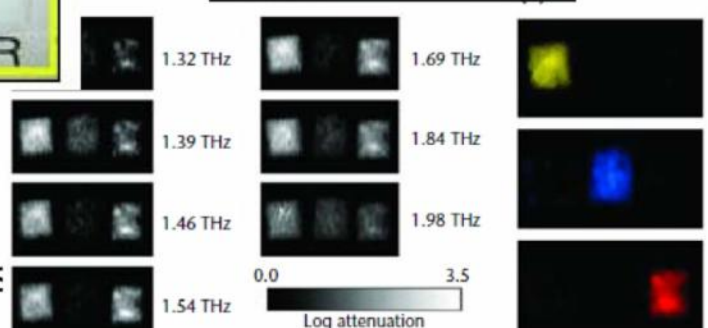
- Why the kinks (antikinks) are relevant to the problem?
- Are there any “Shapiro steps” or “Arnold tongues” ?

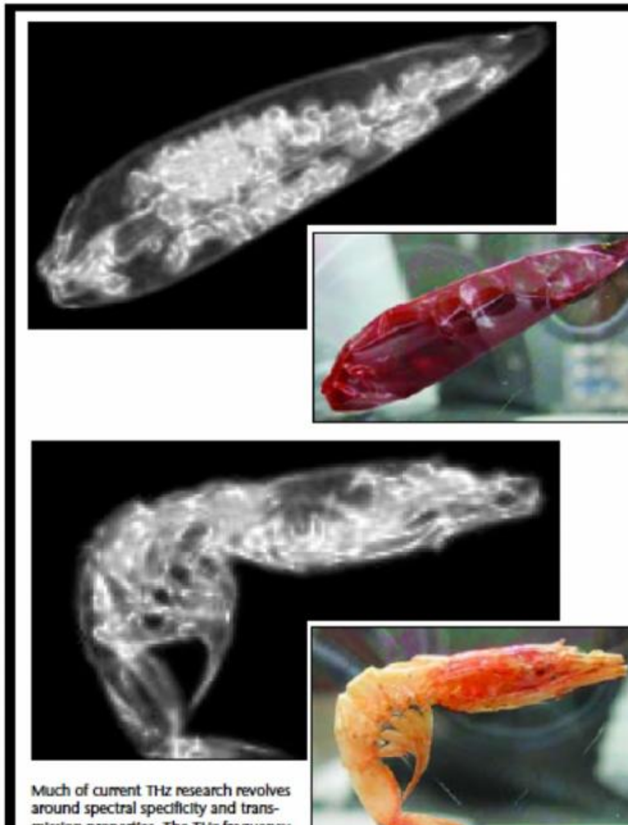

Thank you for attention!

Introduction: Terahertz EM wave

- Important applications range from DNA diagnosis to security check
  - vibration modes of proteins and DNA molecules in the THz range.
- THz imaging







Much of current THz research revolves around spectral specificity and transmission properties. The THz frequency is more selective than X-rays and is therefore more sensitive to the nature of the materials it passes through. Here, a red pepper and a prawn are presented alongside their THz images. [Yuichi Oshima, RIKEN.]

Ref. K. Kawase,
Optics and Photonics News
Oct. 2004, p.38