Differential operators, geodesics and dynamics of localized states on singular spaces.

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Joint work with Vsevolod Chernyshev and Anton Tolchennikov

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Outline

Metric graph — 1D cell complex together with parametrization and metric on edges.

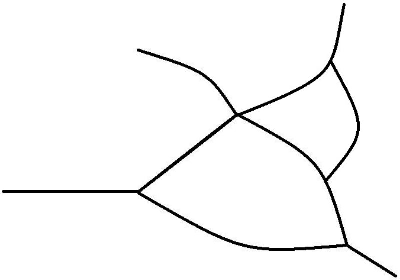
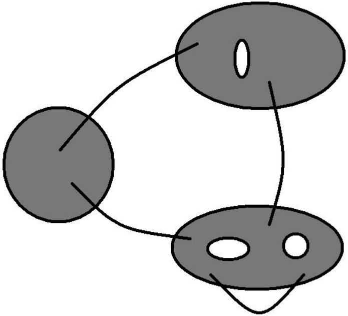


Figure: Graph

Singular spaces

Singular space Γ — topological space, obtained from a graph via replacing vertices by smooth complete Riemannian manifolds M_k, $\mathrm{dim} M_k \leq 3$.



Spectral theory for singular spaces:
M.D. Faddev, B.S. Pavlov, Theor. Math. Phys, 1983, 55(2), 257-268.
B.S. Pavlov, Theor. Math. Phys, 1984, 59(3), 345-353.
P.Exner, P. Sĕba.J. Math. Phys. 28 (1987), 386-391.
J. Brüning and V.A. Geyler, M. Lobanov, S. Roganova, A. Tolchennikov, etc.

Self-adjoint Laplacian

Definition of the Laplace operator $\frac{h^2}{2}\Delta$: 2 conditions.

- $\frac{h^2}{2}\Delta$ is self-adjont;
- If M is a disconnected then

$$\frac{h^2}{2}\Delta = \oplus_j \frac{h^2}{2}\frac{d^2}{dz_j^2}$$

for graphs

$$\frac{h^2}{2}\Delta = \oplus_j \frac{h^2}{2}\frac{d^2}{dz_j^2} \oplus_k \frac{h^2}{2}\Delta_k.$$

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for singular spaces.

Laplacian

Formal definition. Consider the direct sum

$$\frac{h^2}{2}\Delta_0 = \oplus_j \frac{h^2}{2}\frac{d^2}{dz_j^2} \oplus_k \frac{h^2}{2}\Delta_k$$

Definition

$\frac{h^2}{2}\Delta$ is a self-adjoint extension of the restriction $\frac{h^2}{2}\Delta_0|_W$, where

$$W = \{\psi \in Dom(\frac{h^2}{2}\Delta_0), \quad \psi(q_s) = 0\}.$$

Coupling conditions

Boundary conditions: on the edges consider $\psi(q_j)$, $h\psi'(q_j)$. On the manifolds functions have singularities of the Green function type

$$\psi = a_j F(x) + b_j + o(1),$$

$$F = \begin{cases} -\dfrac{1}{h^2\pi}\log\rho, & \dim M = 2; \\ \dfrac{1}{2h^2\pi\rho}, & \dim \mathrm{M} = 3. \end{cases}$$

Coupling conditions

Coupling conditions for graphs.
Vector $\xi = (u, v)$, $u = (h\psi'(q_1), \dots, h\psi'(q_{2E}))$,
$v = (\psi(q_1), \dots, \psi(q_{2E}))$, q_j — endpoints of the edges.
In $\mathbb{C}^{2E} \oplus \mathbb{C}^{2E}$ consider standard skew-Hermitian form

$$< \xi^1, \xi^2 >= \sum_{j=1}^{2E} (u_j^1 \bar{v}_j^2 - v_j^1 \bar{u}_j^2).$$

and fix the Lagrangian (N-dimensional isotropic) plane L.
Coupling conditions

$$\xi \in L, \quad i(E+U)u + (E-U)v = 0,$$

U is unitary matrix.

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Coupling conditions for singular spaces.
Vector $\xi = (u, v)$, $u = (h\psi'(q_1), \ldots, h\psi'(q_{2E}), a_1, \ldots, a_{2E})$,
$v = (\psi(q_1), \ldots, \psi(q_{2E}), b_1, \ldots, b_{2E})$.
In $\mathbb{C}^{4E} \oplus \mathbb{C}^{4E}$ consider standard skew-Hermitian form

$$< \xi^1, \xi^2 > = \sum_{j=1}^{2N} (u_j^1 \bar{v}_j^2 - v_j^1 \bar{u}_j^2).$$

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Local coupling conditions — for each vertex (gluing point) separately:

$$L = \oplus_q L_q$$

.

For each point of gluing q

$\xi_q = (u_q, v_q)$, $u_q = (\psi'(q), a)$, $v_q = (\psi(q), b)$,

$$i(E + U_q)u_q + (E - U_q)v_q = 0,$$

U_q is a unitary 2×2-matrix.

Non-Hermitian Laplacians

Non-Hermitian case: plane L is not Lagrangian (matrix U is not unitary). Then $\xi \in L$ for $\psi \in \mathrm{Dom}(\Delta)$ and $\xi \in L^{\angle}$ for $\psi \in \mathrm{Dom}(\Delta^*)$.

Examples:

- Real Δ (commutes with complex conjugation) $\Leftrightarrow$ real plane L (invariant with respect to complex conjugation)
- Pseudo-Hermitian with respect to complex conjugation: $\overline{\Delta\bar{\psi}} = \Delta^*\psi \Leftrightarrow$ plane L is Lagrangian with respect to the skew symmetric form

$$[\xi^1, \xi^2] = \sum_j (u_j^1 v_j^2 - v_j^1 u_j^2).$$

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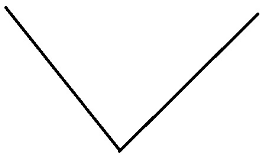
Exotic spectral properties for non-Hermitian case. Example:



If $\psi_1 = \psi_2$, $h\psi_1' = -h\psi_2'$ (self-adjoint case) then $E_n = -(h^2\pi n/2l)^2$.

If $\psi_1 = \psi_2$, $h\psi_1' = h\psi_2'$ (real non-Hermitian case) then $E \in \mathbb{C}$.

Outline

Gaussian packet

Let Γ be a singular space.
Cauchy problem for Schrödinger equation

$$ih\frac{\partial\psi}{\partial t} = -\frac{h^2}{2}\Delta\psi + V\psi$$

$$\psi|_{t=0} = A_0 e^{\frac{iS_0(z)}{h}}.$$

$$S_0 = p_0(z-z_0) + \frac{1}{2}Q_0(z-z_0)^2, \quad \Im p_0 = 0, \quad \Im Q_0 > 0.$$

Asymptotics as $h \to 0$; narrow Gaussian packet (squeezed state) localized on the edge.
Assume that $V = 0$ (Maupertuis principle) and $p_0 = 1$.

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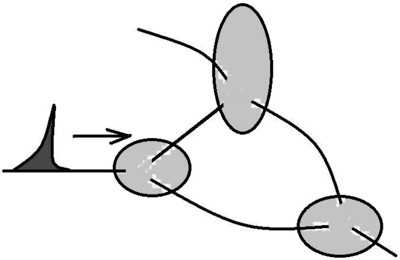

Figure: Initial state

Solution for small t

Asymptotic solution for small t: $\Psi = A(t)e^{\frac{iS(z,t)}{h}}$

$$S(z,t) = \sigma(t) + P(t)(z - Z(t)) + \frac{1}{2}Q(t)(z - Z(t))^2.$$

$(P(t), Z(t))$ is the solution of the classical Hamiltonian system

$$\dot{z} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial z}, \quad H = \frac{1}{2}p^2.$$

$$P(0) = 1, \quad Z(0) = z_0.$$

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Assertion

For sufficiently small t

$$ih\frac{\partial\Psi}{\partial t} = \hat{H}\Psi + O(h^{3/2}).$$

If $\hat{H}$ is self-adjoint, then

$$\psi(z,t,h) = A(t)e^{\frac{iS(z,t)}{h}} + O(\sqrt{h}).$$

Outline

Scattering on a graph

Let Γ be a star graph

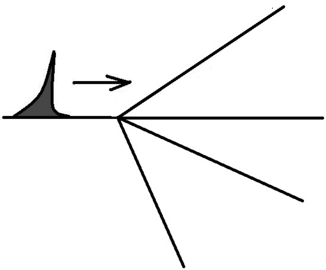

Figure: Scattering on a graph

Theorem

Let t_0 be the time of scattering ($X(t_0) = q$). For certain time interval $t \in (t_0, t_0 + \delta)$ asymptotic solution has the form

$$\Psi = \sum_j A_j(t) e^{\frac{iS_(z,t)}{h}},$$

j-th summand is localized on the j-the edge,

$$\begin{pmatrix} A_1(t_0) \\ A_2(t_0) \\ \dots \\ A_n(t_0) \end{pmatrix} = U \begin{pmatrix} A(t_0) \\ 0 \\ \dots \\ 0 \end{pmatrix},$$

U is the coupling matrix.

Scattering on a manifold

Let Γ be a half-line, connected with the manifold M in a single point q.

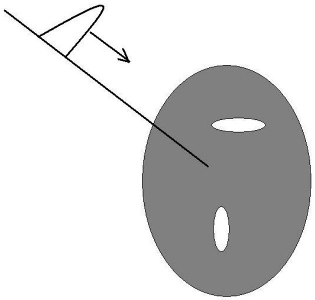

Figure: Scattering on a manifold

Scattering on a manifold

Theorem

For certain time interval asymptotic solution has the form

$$\Psi = \begin{cases} A(t)e^{\frac{iS(z,t)}{h}}, & z \in \mathbb{R}_+, \\ K_{\Lambda_t}[B(x,t)], & x \in M. \end{cases}$$

K is the Maslov canonic operator on the isotropic manifold Λ_t — pull-back of the geodesic sphere — with complex germ. A and B can be expressed explicitly in terms of the coupling matrix U.

Scattering on a manifold

Support of the solution on M.

Consider geodesics, starting from the point of gluing q with unit velocity at the instant of scattering t_0. Their endpoints form the support of the asymptotic solution (the geodesic sphere).

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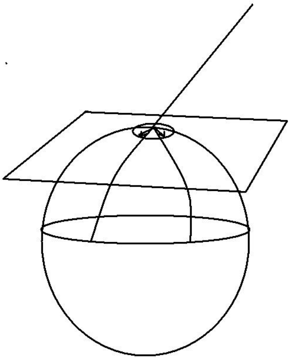

Figure: Support of the solution

Outline

Number of the localized states

Let Γ be a singular space with finite edges. For arbitrary finite t the asymptotic solution has the form

$$\Psi = \sum_j \Psi_j + O(\sqrt{h}),$$

where Ψ_j are localized (Gaussian) packets.
Let $N(t)$ be the number of packets, localized on the edges of Γ (not on the manifolds).
Problem: find the asymptotics of $N(t)$ as $t \to \infty$.
Let l_j be the lengths of the edges and L_j be the lengths of geodesics, connecting gluing points on the manifolds.
Case 1: there is a finite number of lengths $L_1, \dots, L_M$.

Statistics of localized states

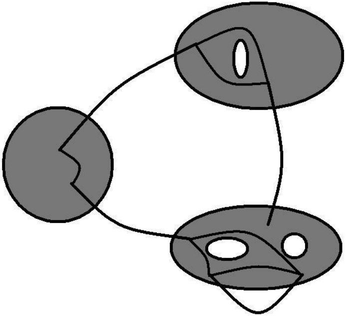

Figure: Lengths of geodesics

Statistics of localized states

Theorem

Let $l_1, \ldots, l_E, L_1, \ldots L_M$ be linearly independent over $\mathbb{Q}$. As $t \to \infty$

$$N(t) = Ct^{M+E-1} + o(t^{M+E-1}).$$

Here E is the number of edges, M is the number of lengths of geodesics L_j.

Constant C

Theorem

For almost all l_j, L_s

$$C = \frac{\sum_{j=1}^{E} l_j}{2^{2E-2}(E+M-1)!\prod_{l=1}^{E} l_j \prod_{s=1}^{M} L_s}.$$

Here E is the number of edges, M is the number of lengths of geodesics L_j.

Polynomial

Theorem

For almost all l_j, L_s

$$N(t) = \sum_{j=1}^{E+M-1} C_j t^j + o(t).$$

Distribution of the packets

Uniform distribution.
Let Δ be a segment on arbitrary edge of Γ.
N_Δ is a number of packets, located on Δ.

Theorem

$$\lim_{t\to\infty}\frac{N_\Delta(t)}{N(t)}=\frac{l_\Delta}{\sum_j l_j}.$$

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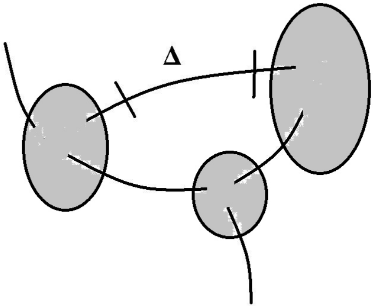


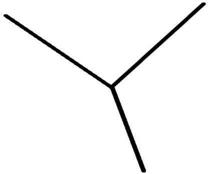
Figure: Uniform distribution

Key point of the proof — reduction to the certain problem of analytic number theory — counting the number of integral points in a large polyhedron. Let Q be a polyhedron in $\mathbb{R}^N$ and let Q_t be a t-homothetic polyhedron. Let $m(t)$ be the number of lattice points in Q_t. General problem: what is the asymptotics of $m(t)$ as $t \to \infty$?

Hardy

Example: Star with three edges — Hardy - Littlewood formula (1927).

$$N(t) = \frac{t^2}{8}\frac{l_1 + l_2 + l_3}{l_1 l_2 l_3} + \frac{t}{2}\left(\frac{1}{l_1} + \frac{1}{l_2} + \frac{1}{l_3}\right) + o(t).$$



Outline

The cylinder with the segment

Examples of spaces with infinite number of geodesics.
1. Flat cylinder with the segment glued in two points.

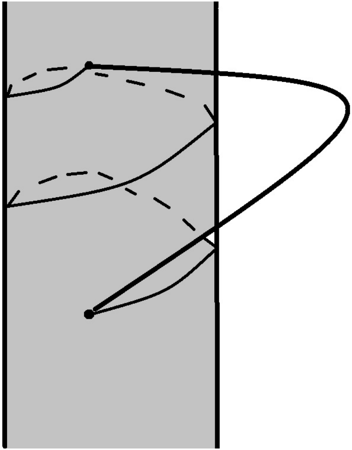


Figure: The segment glued to the cylinder in two points

$$N(t) = \exp\left(\sqrt{\frac{2}{3R}}\pi\sqrt{t}(1 + o(1))\right).$$

Here R is the radius of the cylinder.
This is the Hardy — Ramanujan formula for the number of partitions of the integer t in integer summands.

Flat torus with the segment

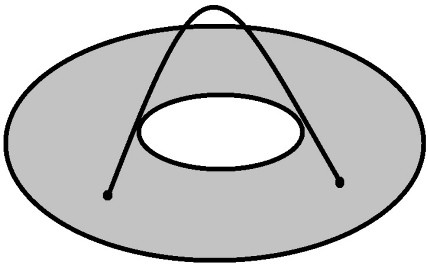


Figure: The segment glued to the flat torus in two points

$$N(t) = \exp\left(3\sqrt[3]{\frac{5\pi}{8ab}\zeta(3)}t^{2/3}(1+o(1))\right).$$

Here a, b are lengths of the fundamental cycles of the torus. For 3D torus

$$N(t) = \exp\left(4\sqrt[4]{\frac{\pi}{3abc}\zeta(4)}t^{3/4}(1+o(1))\right).$$

General result

Let $m(t)$ be the total number of geodesics, connecting pairs of the points of gluing, s.t. their lengths are at most t. **Case 2: suppose that $m(t) = \alpha t^{\gamma}(1 + t^{-\epsilon})$.**

Theorem

Let S be the total set of the lengths of geodesics. Let there exists a finite subset $S_0 \subset S$, s.t. the set $S \backslash S_0$ is linearly independent over $\mathbb{Q}$. Then

$$\log N(t) = (\gamma + 1)\left(\frac{\alpha\Gamma(\gamma+1)\zeta(\gamma+1)}{\gamma^{\gamma}}\right)^{\frac{1}{\gamma+1}} t^{\frac{\gamma}{\gamma+1}}(1 + o(1))$$

Outline

Case 3: suppose that $m(t) = \mathrm{e}^{Ht(1+O(t^{-\epsilon}))}$ (usually — positive topological entropy).

Topological entropy

Let M be a Riemannian manifold, $m_{x,y}(t)$ — number of geodesics, connecting pair of points $x, y \in M$, s.t. their lengths are at most t. Then

$$\mathcal{H} = \lim_{t\to\infty} \frac{1}{t} \int_{M\times M} m_{x,y}(t)dxdy$$

is the topological entropy of the geodesic flow. For manifolds without conjugate points

$$\mathcal{H} = \lim_{t\to\infty} \frac{1}{t} m_{x,y}(t) \quad \forall x, y \in M.$$

Theorem

$$\log N(t) = H(1 + o(1)).$$

Key point of the proof — reduction to the certain problem of analytic number theory — namely, the problem of abstract primes.

Abstract primes

Arithmetical semigroup $G = \oplus_{j \in J} \mathbb{Z}_+$, J - countable.
Homomorphism $\rho : G \to \mathbb{R}_+$, we identify j with the generator of $\mathbb{Z}_+$.

$$m(t) = \sharp\{j \in J | \rho(j) \leq t\}, \quad N(t) = \sharp\{g \in G, \rho(g) \leq t\}.$$

If one knows the asymtotics of m (N), how to compute the asymptotics of N (m)?
j—primes,$\rho(j)$=log j: $m(\log t)$ — distribution of primes,
j—integers, $\rho(j) = j$: $N(t)$ — number of partitions of integer t.

Open questions

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- What happens for commensurable lengths? Conjecture: the number of lengths must be replaced by the rang of the corresponding set over $\mathbb{Q}$ (proved in certain special cases by V.L. Chernyshev and A.A. Tolchennikiov).
- If the rang is 1 (for metric graph), the number $N(t)$ is constant for $t \geq T$. How to compute T? First examples show, that T is connected with the Frobenius numbers.
- How does the energy (probability of finding a particle on the certain edge) behave? Almost nothing is known; for the wave equation certain concrete situations were studied by A.V. Tsvetkova.

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THANK YOU FOR YOUR ATTENTION!