Deformation quantization and vector fields

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Let (M, π) be a Poisson manifold, i.e. M — smooth closed manifold, $\pi \in \Lambda^2(TM)$ a bivector, such that

$$[\pi, \pi] = 0$$

where $[,]$ denotes the Schouten bracket on M: the unique graded Lie bracket, verifying Leibniz rule and Jacobi identity and such coinciding with commutator on vector fields. In this case the bracket on $C^\infty(M)$

$$\{f, g\} = \pi(df, dg)$$

is Poisson bracket.

Examples:

1. Symplectic manifolds, e.g. $\mathbb{R}^{2n}$ with standard structure
2. $M = \mathfrak{g}^*$, and π — Kirillov-Poisson structure, Poisson bracket given by

$$\{f, g\}(\xi) = \xi([df, dg]).$$

Deformation quantization of Poisson manifold: a product on the formal power series

$$f * g = fg + \frac{\hbar}{2}\{f, g\} + \sum_n \frac{\hbar^n}{2} B_n(f, g).$$

Here $f, g \in C^\infty(M)$, $\hbar$ — formal variable, B_k are suitable bidifferential operators and $*$ is a $\hbar$-linear product in $\mathcal{A} = C^\infty(M)[[\hbar]]$.
Example: $M = \mathbb{R}^{2n}, \pi = \pi^{ij}\partial_i \otimes \partial_j,\ \pi^{ij} = const$, we put

$$f * g = \exp(\pi^{ij}\overleftarrow{\partial_i} \otimes \overrightarrow{\partial_j})(f \otimes g),$$

the Moyal product.
Question (Flato): is it possible to find a continuation of the power series above for any Poisson structure?

Answer to this question: Yes!
A bit of history: first, in the beginning of 1980-ies this was proved by Gutt, Lecomt, de Wilde for symplectic manifolds, later, in the end of 80-ies another (easier and more explicit) construction was given by Fedosov; finally, in 1997 M.Kontsevich proved the general result. We shall discuss his construction later.
Now we are interested in the following question, related with the question: what other structures can be transferred from classical to quantized side?
An important structure: integrable systems, i.e. commutative subalgebras (with respect to Poisson structure). Is it possible to extend them to commutative subalgebras in $\mathscr{A}$? If yes, this is called "quantum integrable systems".

In 2012 in a joint work with D. Talalaev we addressed this question. Let me briefly recall the results:

Theorem

Let C be a commutative subalgebra in the Poisson algebra on M; then there exists a sequence of characteristic classes in the Hochschild cohomology of C with coefficients in $C^\infty(M)$, which vanish iff there exists an extension of C to a commutative subalgebra in $\mathcal{A}$.

This series of obstructions is hard to compute. Sometimes we can facilitate the process:

Proposition

Let $C = \mathbb{R}[f_1, \dots, f_k]$. If the map $M \to \mathbb{R}^k$ is regular submersion, then one can replace the obstructions above by classes in a Poisson cohomology complex of $\mathbb{R}^k$ with coefficients in $C^\infty(M)$: $CP^* = C^\infty(M) \otimes \Lambda^*\mathbb{R}^k$, with differential

$$d_\pi(f \otimes v) = \sum_i \{f_i, f\} \otimes e^i \wedge v.$$

In general finding a commutative subalgebra is a big deal, a matter of art (I mean, in classical case). But sometimes there exist simple and regular methods to generate commutative subalgebras. One of the most popular methods of this sort is the “argument shift method”.

The simplest way to explain it is as follows: we shall call $\xi \in Vect(M)$ “Nijenhuis field”, if $L^2_\xi(\pi) = 0$. In this case: put $\pi_\xi = L_\xi(\pi) \neq 0$, then

$$[\pi_\xi, \pi] = [\pi_\xi, \pi] = 0,$$

i.e. π, π_ξ is a matching pair of Poisson structures. Let f, g be any Casimir functions on M, i.e. such functions, that

$$\{f, h\} = \{g, h\} = 0$$

for all functions $h \in C^\infty(M)$. Then

$$\{\xi^k f, \xi^l g\} = 0$$

for all k, l.

So, we need to transfer vector fields from classical to quantum case!
Let us begin doing it step by step: put

$$\Xi = \xi + \hbar\xi_1 + \dots$$

and plug this formula into the Leibniz rule for the $*$-product. We shall see at once that $L_\xi \pi$ should be equal to 0. Such vector fields are called "Poisson fields".

If we continue further, we obtain the following result:

Proposition

There exists a series of obstructions in the Poisson-Lichnerowicz cohomology of M, which vanish, if there exist a way to extend the series above to a differentiation of $\mathcal{A}$.

Observe, that in fact, these obstructions depend very much on the method we use to extend the series at each stage: in particular, they show rather, if it is possible to extend any given first n terms of the series to a differentiation. Later I shall explain, why in principle, at least one extension always exists.

Let us now consider several Poisson fields, spanning a Lie algebra $\mathfrak{g}$, acting on M. The question is, if it is possible to extend the series for all the fields in $\mathfrak{g}$so that we obtain an action of $\mathfrak{g}$on $\mathscr{A}$. One more time we can work rather naively and obtain the following result:

Proposition

There exists an extension of the Poisson action of $\mathfrak{g}$on M to an action by derivations on $\mathscr{A}$, if certain classes in the Lie algebra cohomolgy of $\mathfrak{g}$with values in Lichnerowicz-Poisson complex of M vanish.

Once again this result answers the question, whether certain formal extensions of the vector fields to power series in $\hbar$ can be continued indefinitely. Also remark, that Lichnerowicz-Poisson cohomology in case when M is symplectic coincides with de Rham cohomology.

We can apply the same technic to the question, whether one can extend a Nijenhuis vector field to a Nijenhuis operator on $\mathcal{A}$. Here we say that an operator $N : \mathcal{A} \to \mathcal{A}$ is Nijenhuis, if

$$L_N^2(*) = 0,$$

where

$$L_N(*)(f, g) = N(f * g) - N(f) * g - f * N(g).$$

If N is Nijenhuis, then one can repeat the reasoning for the Nijenhuis field and obtain the following result: if F, G are the elements of the center of $\mathcal{A}$, then

$$[N^k(F), N^l(G)] = 0$$

for all k, l.

One knows that every Casimir in $C^\infty(M)$ can be extended to an element in the center of $\mathcal{A}$(certain explanation to be given later). So the purpose is to use these elements and Nijenhuis operator to obtain commutative subalgebras.

Thus, suppose that ξ is a Nijenhuis vector field, and consider a series

$$N = \xi + \hbar\xi_1 + \hbar^2\xi_2 + \dots$$

The condition that $L_N^2(*) = 0$ gives a series of equations of the form:

$$[\delta\xi_{n+1}, \xi] = \sum_{p+q+r=n+1} [[B_p, \xi_q], \xi_r].$$

Here δ is the Hochschild differential. We can interpret the term on the left as the value of the ξ-twisted Hochschild differential on ξ_{n+1}: $\delta_\xi(\varphi) = [\delta\varphi, \xi]$, then $\delta_\xi^2 = 0$.

Let $H^*_\xi(C^\infty(M))$ denote the corresponding cohomology. It is very hard to compute it! For instance, if $M = \mathfrak{g}^* = \mathbb{R}^n$, and $\xi = \frac{\partial}{\partial x_1}$ a constant field, we shall have

Proposition

The twisted Hochschild cohomology of the algebra $A = \mathbb{R}[x_1, \ldots, x_n]$ with respect to the vector field ∂_1 is equal to the direct sum:

$$H^*_{\partial_1}(\mathcal{A}) = \mathcal{T}^*_{>0}(\mathbb{R}^n) \oplus \left(C^*_0(A)/\mathrm{Im}\,\delta\right),$$

where $\mathcal{T}^*_{>0}(\mathbb{R}^n)$ denotes the space of polynomial polyvector fields on $\mathbb{C}^n$, non-constant in direction of x_1 and $C^*_0(A)$ denote the subcomplex of Hochschild cochains, independent of x_1.

Thus, we are obliged to resort to more powerful tools. One of these tools is the L_∞-map of Kontsevich: one says, that an L_∞-morphism F between two DG Lie algebras $\mathfrak{g}_1$ to $\mathfrak{g}_2$ is given, if there is a collection of maps $F_n : \Lambda^n\mathfrak{g}_1 \to \mathfrak{g}_2$ of degrees $1-n$, which verify the following sequence of equations (here we omit the signs of $\wedge$-product):

$$
\begin{aligned}
&dF_{n+1}(X_1, X_2, \dots, X_{n+1})\\
&= \sum_{1\le i<j\le n+1} (-1)^{\epsilon(i,j)} F_n([X_i, X_j]_1, X_1, \dots, \widehat{X_i}, \dots, \widehat{X_j}, \dots, X_{n+1})\\
&\quad + \frac{1}{2}\sum_{i=1}^{n}\sum_{\sigma\in S_{n+1}} \frac{(-1)^{\sigma(X)}}{i!(n-i+1)!}[F_i(X_{\sigma(1)}, \dots, X_{\sigma(i)}), F_{n-i+1}(X_{\sigma(i+1)}, \dots, X_{\sigma(n+1)})]_2.
\end{aligned}
$$

Here as usually $\widehat{\ }$ denotes missing element, S_{n+1} is the group of permutations in $n+1$ elements and the signs are obtained from Koszul sign rules and depend on the degrees of the elements.

Kontsevich constructed such map from $\Lambda^* TM$ to the Hochschild complex of $C^\infty(M)$. Then, if we apply it, we obtain:

$$F_\pi(\xi) = \sum_n \frac{\hbar^n}{n!} F_{n+1}(\xi, \pi, \dots, \pi),$$

is a derivation of $\mathcal{A}$, if ξ is a Poisson vector field.
However, this method does not work for Lie algebra actions: we shall have

$$[F_\pi(X), F_\pi(Y)] - F_\pi([X, Y]) = ad_{\Phi(X,Y)},$$

where $ad_f,\ f \in C^\infty(M)[[\hbar]]$ is the inner derivative of the deformed algebra $(C^\infty(M)[[\hbar]], *)$ with respect to f, and

$$\Phi(X, Y) = \sum_n \frac{\hbar^n}{n!} F_{n+2}(X, Y, \pi, \dots, \pi).$$

It is this extra term, that gives obstructions in previous method.