Polynomial forms for quantum Calogero-Moser Hamiltonians and commutative subalgebras in universal enveloping algebras

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Universal enveloping algebra of $gl(n)$

The universal enveloping algebra $U(gl_n)$ is generated by elements e_{ij} and relations $e_{ij}e_{kl} - e_{kl}e_{ij} = \delta_{jk}e_{il} - \delta_{i,l}e_{kj}$.

The most known representation by differential operators has the form

$$e_{ij} \to x_i \frac{\partial}{\partial x_j}.$$

One more representation $e_{i,j} \to E_{i-1,j-1}$, where

$$E_{ij} = y_i \tfrac{\partial}{\partial y_j}, \qquad E_{0i} = \tfrac{\partial}{\partial y_i},$$
$$E_{00} = -\textstyle\sum_{j=1}^{n-1} y_j \tfrac{\partial}{\partial y_j} + \beta\, n, \qquad E_{i0} = y_i E_{00}$$

will be used in my talk. Notice that if $k = \beta n$ is a natural number than these operators preserve the space of polynomials of degrees $\leqslant k$.

A reasonable state of a classification problem for integrable quantum systems is: a classification of (maximal) commutative subalgebras in $U(gl_n)$.

We claim that the quantum elliptic Calogero-Moser Hamiltonians with n particals are generated by proper commutative maximal subalgebras in $U(gl_n)$.

Classical limit

Suppose that two elements f and g of $U(gl_n)$ commute. If we regards e_{ij} as commuting variables and take symbols $\bar{f}$ and $\bar{g}$ of f and g, then

$$\{\bar{f}, \bar{g}\} = 0,$$

where $\{,\}$ is the linear Poisson bracket that corresponds to the Lie algebra gl_n.

In the classical case we have

$$E_{ij} = q_i p_j, \qquad E_{0i} = p_i,$$

$$E_{00} = -\sum_{j=1}^{n-1} q_j p_j + \beta, \qquad E_{i0} = q_i\, E_{00}.$$

One can verify that this is a Darboux coordinates on the minimal symplectic leave of the gl_n-Poisson bracket. This leave is the orbit of the diagonal matrix $\mathrm{diag}(\beta, 0, 0, ..., 0)$, i.e. the set of all rank one matrices U with $\mathrm{trace}\,\mathrm{U} = \beta$.

Conjecture. There exists a quadratic Poisson bracket compatible with the linear gl_n-Poisson bracket such that the bi-Hamiltonian integrals of the pencil restricted on the minimal symplectic leave of the gl_n-bracket coincides with the polynomial forms of the classical Calogero-Moser Hamiltonians.

Polynomial forms

We consider the elliptic quantum A_N-Calogero-Moser Hamiltonians

$$H_N = -\Delta + \beta(\beta - 1) \sum_{i \neq j}^{N+1} \wp(x_i - x_j), \tag{1}$$

where $\Delta = \sum_i \frac{\partial^2}{\partial x_i^2}$, β is an arbutrary parameter, and $\wp(x)$ is the Weierstrass function with the invariants $g_2,\ g_3$.

In the coordinates

$$X = \frac{1}{n+1}\sum_{i=1}^{n+1} x_i, \qquad y_i = x_i - X.$$

the Hamiltonian becomes

$$H_n = -\frac{1}{n+1}\frac{\partial^2}{\partial X^2} + \mathcal{H}_n\left(y_1, y_2, \ldots y_n\right),$$

where

$$\mathcal{H}_n = -\frac{n}{n+1}\sum_{i=1}^{n}\frac{\partial^2}{\partial y_i^2} + \frac{1}{n+1}\sum_{i\neq j}^{n}\frac{\partial^2}{\partial y_i \partial y_j} + \beta(\beta-1)\sum_{i\neq j}^{n+1}\wp(y_i - y_j).$$

In the potential we substitute $y_{n+1} = -\sum_{i=1}^{n} y_i$.

A class of integrable Hamiltonians

$$H = \Delta + U(x_1, ..., x_n),$$

related to simple Lie algebras. For such Hamiltonians the potential U is a rational, trigonometric or elliptic function.

Observation 1 (A.Turbiner). For many of these Hamiltonians there exists a change of variables and a gauge transformation that bring the Hamiltonian to a differential operator with polynomial coefficients.

Example. Consider the Calogero model with $n = 2$:

$$H = \Delta + g \sum_{i>j}^{3} \frac{1}{(x_i - x_j)^2}.$$

Let $Y = \sum_{i=1}^{3} x_i$ and $y_i = x_i - \frac{Y}{3}$. Then

$$\Delta = -3\frac{\partial^2}{\partial Y^2} - \frac{2}{3}\Big(\frac{\partial^2}{\partial y_1^2} + \frac{\partial^2}{\partial y_2^2} - \frac{\partial^2}{\partial y_1 \partial y_2}\Big).$$

Thus we have reduced the Hamiltonian to the following two dimensional one:

$$\mathcal{H} = -\frac{1}{3}\Big(\frac{\partial^2}{\partial y_1^2} + \frac{\partial^2}{\partial y_2^2} - \frac{\partial^2}{\partial y_1 \partial y_2}\Big) + \beta(\beta - 1) \sum_{i>j}^{3} \frac{1}{(y_i - y_j)^2}. \quad (2)$$

Here $y_3 = -y_1 - y_2$.

The change of variables

$$x = -y_1^2 - y_2^2 - y_1 y_2, \qquad y = -y_1 y_2 (y_1 + y_2)$$

and the gauge transformation $\mathcal{H} \to h^{-1} \mathcal{H} h$, where

$$h = (x - y)^\beta (2x + y)^\beta (x + 2y)^\beta,$$

bring $\mathcal{H}$ to the polynomial form

$$L = x \frac{\partial^2}{\partial x^2} + 3y \frac{\partial^2}{\partial x \partial y} - \frac{1}{3} x^2 \frac{\partial^2}{\partial y^2} + (1 + 3\beta) \frac{\partial}{\partial x}. \quad \square$$

In the trigonometric case the transformation to a polynomial form is given by

$$x = \cos y_1 + \cos y_2 + \cos(y_1 + y_2) - 3,$$

$$y = \sin y_1 + \sin y_2 - \sin(y_1 + y_2).$$

Recently (**A.Turbiner, VS**) the transformation

$$x = \frac{\wp'(y_1) - \wp'(y_2)}{\wp(y_1)\wp'(y_2) - \wp(y_2)\wp'(y_1)}, \quad y = \frac{\wp(y_1) - \wp(y_2)}{\wp(y_1)\wp'(y_2) - \wp(y_2)\wp'(y_1)}$$

that brings the elliptic Calogero-Moser Hamiltonian to a polynomial form has been found.

This transformation has been found by a deformation of the above transformation for the trigonometric model.

Conjecture 1. The analog of the above transformation for arbitrary n is given by the following system of linear equations

$$M\mathbf{x} = \mathbf{e},$$

where $\mathbf{x} = (x_1, \ldots, x_n)^t$, $\mathbf{e} = (1, 1, \ldots, 1)^t$ with

$$M_j^i = \frac{d^{j-1}\wp(y_i)}{dy_i^{j-1}} \ .$$

This formula gives correct transformations in the cases $n = 1, 2, 3$. The corresponding polynomial form in the case $n = 3$ is new.

It is easy to verify that the operators

$$E_{ij} = y_i \frac{\partial}{\partial y_j}, \quad E_{0i} = \frac{\partial}{\partial y_i},$$

$$E_{00} = -\sum_{j=1}^{n} y_j \frac{\partial}{\partial y_j} + \beta(n+1), \qquad E_{i0} = y_i E_{00},$$

where $i, j = 1, \ldots, n$, satisfy the commutator relations

$$[E_{ij}, E_{kl}] = \delta_{jk} E_{il} - \delta_{i,l} E_{kj},$$

and, therefore, define a representation of the Lie algebra gl_{n+1} and universal enveloping algebra $U(gl_{n+1})$. This representation is not exact.

Notice that if $k = -\beta(n+1)$ is a natural number than these operators preserve the space of polynomials of degrees $\leqslant k$.

If we replace the operators E_{ii} by

$$H_1 = -\sum_{j=1}^{n} y_j \frac{\partial}{\partial y_j} - y_1 \frac{\partial}{\partial y_1} - \beta(n+1), \quad H_i = y_{i-1} \frac{\partial}{\partial y_{i-1}} - y_i \frac{\partial}{\partial y_i},$$

$i = 2, \ldots n$, these generators define representations of sl_{n+1} and $U(sl_{n+1})$.

Conjecture 2. The polynomial form P_n can be written as a linear combination of anti-commutators of the operators E_{ij} and H_i, where $i \neq j,\ i, j = 1$.

The case $n = 1$.

In the simplest case $n = 1$ the elliptic Calogero-Moser Hamiltonian is just the Lame operator

$$H = -\frac{1}{2}\frac{\partial^2}{\partial y_1^2} + 2\beta(\beta - 1)\,\wp(2y_1).$$

It is well known that the gauge $\mathcal{H}_1 \to D^{-\beta}\mathcal{H}_1 D^{\beta}$, where

$$D = \frac{\wp'(y_1)}{\wp(y_1)^2},$$

and the change of variables

$$u = \frac{1}{\wp(y_1)}$$

bring the Lame operator to a polynomial form.

It turns out that this polynomial form P_1 can be written as

$$P_1 = \frac{1}{2}\{H, E\} + \frac{g_2}{8}\{H, F\} + \frac{g_3}{4}\{F, F\},$$

where $\{*, *\}$ means the anti-commutator and

$$H = -(2u\frac{\partial}{\partial u} + 2\beta), \qquad E = \frac{\partial}{\partial u}, \qquad F = -u(u\frac{\partial}{\partial u} + 2\beta).$$

These operators satisfy the sl_2-commutator relations

$$[H, E] = 2E, \qquad [H, F] = -2F, \qquad [E, F] = H,$$

and therefore, P_1 is an element of the representation of the universal enveloping algebra for sl_2.

It can be easily verified that any second order differential operator P generated by E, F, G has the form

$$P = (a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0)\frac{d^2}{dx^2} +$$

$$(b_3x^3 + b_2x^2 + b_1x + b_0)\frac{d}{dx} + c_2x^2 + c_1x + c_0,$$

where the coefficients are related by the following identities

$$b_3 = 2(1-k)\,a_4, \qquad c_2 = k(k-1)\,a_4, \qquad c_1 = k(a_3 - ka_3 - b_2).$$

The transformation group

$$x \to \frac{s_1 x + s_2}{s_3 x + s_4}, \qquad P \to (s_3 x + s_4)^{-k} P (s_3 x + s_4)^k, \qquad (3)$$

acts on the nine-dimensional vector space of such operators. The coefficient $a(x)$ at the second derivative is a fourth order polynomial which transforms as follows

$$a(x) \to (s_3 x + s_4)^4 a\left(\frac{s_1 x + s_2}{s_3 x + s_4}\right).$$

If $a(x)$ has four distinct roots, we call the operator P *elliptic*. In the elliptic case using transformations (3), we may reduce a to

$$a(x) = 4\,x(x-1)(x-\kappa).$$

Define parameters $n_1, ..., n_5$ by identities

$$b_0 = 2(1+2n_1), \quad b_1 = -4\Big((\kappa+1)(n_1+1) + \kappa n_2 + n_3\Big),$$

$$b_2 = -2\,(3 + 2n_1 + 2n_2 + 2n_3),$$

$$k = -\frac{1}{2}(n_1 + n_2 + n_3 + n_4),$$

$$n_5 = c_0 + n_2(1-n_2) + \kappa n_3(1-n_3) + (n_1+n_3)^2 + \kappa(n_1+n_2)^2.$$

Then the operator $H = hPh^{-1}$, where

$$h = x^{\frac{n_1}{2}}(x-1)^{\frac{n_2}{2}}(x-\kappa)^{\frac{n_3}{2}}$$

has the Laplace-Beltrami form

$$H = a(x)\frac{d^2}{dx^2} + \frac{a'(x)}{2}\frac{d}{dx} + n_5 + n_4(1-n_4)\,x + \frac{n_1(1-n_1)\kappa}{x} +$$

$$\frac{n_2(1-n_2)(1-\kappa)}{x-1} + \frac{n_3(1-n_3)\kappa(\kappa-1)}{x-\kappa}.$$

When

$$k = -\frac{1}{2}(n_1 + n_2 + n_3 + n_4)$$

is a natural number, the operator H preserves the finite-dimensional polynomial vector space V_k.

Now after the transformation $y = f(x)$, where

$$f'^2 = 4f(f-1)(f-\kappa)$$

we arrive at

$$H = \frac{d^2}{dy^2} + n_5 + n_4(1-n_4)\,f + \frac{n_1(1-n_1)\kappa}{f} +$$

$$\frac{n_2(1-n_2)(1-\kappa)}{f-1} + \frac{n_3(1-n_3)\kappa(\kappa-1)}{f-\kappa}.$$

In general here n_i are arbitrary parameters.

Another form of this Hamiltonian (up to a constant) is given by

$$H = \frac{d^2}{dy^2} + n_4(1-n_4)\,\wp(y) + n_1(1-n_1)\,\wp(y+\omega_1) +$$

$$n_2(1-n_2)\,\wp(y+\omega_2) + n_3(1-n_3)\,\wp(y+\omega_1+\omega_2),$$

where ω_i are half-periods of the Weierstrass function $\wp(x)$. If $n_1 = n_2 = n_3 = 0$ we get the Lame operator. In general, it is the Darboux-Treibich-Verdier operator.

Rational degeneration for arbitrary n.

Let us verify Conjecture 1 for the rational degeneration $\wp(y) \to \frac{1}{y^2}$. In this case the general formula for the transformation yields

$$\begin{pmatrix} y_1^{n-1} & y_1^{n-2} & \dots & y_1 & 1 \\ y_2^{n-1} & y_2^{n-2} & \dots & y_2 & 1 \\ \vdots & \vdots & \dots & \vdots & \vdots \\ y_n^{n-1} & y_n^{n-2} & \dots & y_n & 1 \end{pmatrix} \begin{pmatrix} u_1 \\ -2!\, u_2 \\ \vdots \\ (-1)^{n+1} n!\, u_n \end{pmatrix} = \begin{pmatrix} y_1^{n+1} \\ y_2^{n+1} \\ \vdots \\ y_n^{n+1} \end{pmatrix}$$

and

$$D_n = \prod_{i>j}^{n+1} (y_i - y_j)^2.$$

The solution can be written can be written in terms of the Schur polynomials

$$u_k = \frac{1}{k!}\; s_{\underbrace{(2,1,1\dots,1,1)}_{k}} \left(y_1, y_2, \dots y_n\right), \qquad k = 1, \dots, n.$$

It is easy to see that

$$s_{\underbrace{(2,1,1\ldots,1,1)}_{k}}(y_1, y_2, \ldots y_n) = -\sigma_k(y_1, y_2, \ldots y_{n+1}),$$

where

$$\sigma_k(x_1, x_2, \ldots x_m) = \sum_{1<i_1<i_2<\cdots<i_k<m} x_{i_1} x_{i_2} \ldots x_{i_m}$$

are elementary symmetric polynomials. Thus we get the transformation $u_k = -\frac{1}{k!}\sigma_k(y_1, y_2, \ldots y_{n+1})$ that coincides with one found by Rühl and Turbiner up to the multipliers $-\frac{1}{k!}$.

Obviously the polynomial form is not unique. If we take the power sums $p_n = \sum_i x_i^n$ instead of elementary symmetric polynomials, we obtain the following known formula:

$$\tilde{H} = \sum_{k,n\geqslant 1} knp_{n+k-2}\frac{\partial}{\partial p_k}\frac{\partial}{\partial p_n} + (1-\beta)\sum_{n>1} n(n-1)p_{n-2}\frac{\partial}{\partial p_n} +$$

$$\beta\sum_{n,k\geqslant 0}(k+n+2)p_k p_n\frac{\partial}{\partial p_{k+n+2}}.$$

One should replace p_n and $\frac{\partial}{\partial x_n}$ by zero for $n > n+1$.

Remarkable quadratic element in $U(gl_3)$.

Consider the universal enveloping algebra $U(gl_3)$ generated by elements e_{ij} and relations $e_{ij}e_{kl} - e_{kl}e_{ij} = \delta_{jk}e_{il} - \delta_{i,l}e_{kj}$. It turns our that the element

$$H = h_0 + h_1 g_2 + h_2 g_2^2 + h_3 g_3, \tag{4}$$

where

$$h_0 = 12e_{12}e_{11} - 12e_{32}e_{13} - 12e_{33}e_{12} - e_{23}^2,$$

$$h_1 = -e_{21} + 2e_{21}e_{11} - e_{22}e_{21} - e_{31}e_{23} - 12e_{32}^2 - e_{33}e_{21},$$

$$h_2 = -e_{31}^2, \qquad h_3 = 36e_{32}e_{31} + 3e_{21}^2,$$

commutes with two third order elements of the form

$$K = K_0 + K_1 g_2 + K_2 g_3,$$

$$L = L_0 + L_1 g_2 + L_2 g_3 + L_3 g_2^2 + L_4 g_2 g_3 + L_5 g_3^2 + L_6 g_2^3.$$

One can verify that $[K, L] = 0$.

Different representations of $U(gl_3)$ by differential, difference and q-difference operators generate "integrable"operators. In particular, the representation by

$$E_{ij} = y_i \frac{\partial}{\partial y_j}, \quad E_{0i} = \frac{\partial}{\partial y_i},$$

$$E_{00} = -\sum_{j=1}^{n} y_j \frac{\partial}{\partial y_j} + \beta(n+1), \qquad E_{i0} = y_i E_{00}$$

maps the element H to a polynomial form P_2 for A_2-Calogero-Moser Hamiltonian, the element L to a third order differential operator that commutes with P_2, and K to zero.

Notice that the representation of $U(gl_3)$ by the matrix unities in Mat_3 maps H, K and L to zero.

Consider the representation defined by

$$e_{ij} \to x_i \frac{\partial}{\partial x_j}.$$

It maps H to a homogeneous differential operator with 3 independent variables of the form $H_2 = \sum_{i \geqslant j} a_{ij} \frac{\partial^2}{\partial x_i \partial x_j}$, where

$$a_{11} = -2g_2 x_1 x_2 - 3g_3 x_2^2 + g_2^2 x_3^2, \qquad a_{22} = 12 g_2 x_3^2,$$

$$a_{33} = x_2^2, \qquad a_{21} = -12 x_1^2 + g_2 x_2^2 - 36 g_3 x_3^2,$$

$$a_{31} = 2\, g_2 x_2 x_3, \qquad a_{32} = 24\, x_1 x_3.$$

and L to an operator $L_2 = \sum_{i \geqslant j \geqslant k} b_{ijk} \frac{\partial^3}{\partial x_i \partial x_j \partial x_k}$. It is interesting that in both H_2 and in L_2 the lower order terms are absent.

In the classical limit we get functions $I_2 = \sum_{i \geqslant j} a_{ij}\, p_i p_j$ и $I_3 = \sum_{i \geqslant j \geqslant k} b_{ijk}\, p_i p_j p_k$, which commute with respect to the standard Poisson bracket. Since there exists a trivial integral of notion $I_1 = \sum x_i p_i$, the quadratic Hamiltonian is Liouville integrable.

Although the Hamiltonian should be somehow related to the classic elliptic Calogero-Moser model it possesses some intriguing properties and could be interesting to experts in integrable cases in Hamiltonian mechanics.

The case of $U(gl_4)$.

The element $H \in U(gl_4)$ has the form

$$H = h_0 + h_1\, g_2 + h_2\, g_3 + h_3\, g_2^2 + h_4\, g_2 g_3,$$

where

$$h_0 = 36\, e_{12}e_{11} + 12\, e_{22}e_{12} - 3\, e_{23}^2 -$$

$$e_{24}e_{11} - e_{24}e_{22} - 24\, e_{32}e_{13} - 12\, e_{33}e_{12} - e_{33}e_{24} -$$

$$2\, e_{34}e_{23} - e_{34}^2 - 24\, e_{42}e_{14} - 72\, e_{43}e_{13} - 36\, e_{44}e_{12} + 3\, e_{44}e_{24},$$

$$h_1 = -4\,e_{21} + 5\,e_{21}e_{11} - e_{22}e_{21} - 2\,e_{31}e_{23} - 24\,e_{32}^2 -$$
$$e_{33}e_{21} - 2\,e_{41}e_{24} - 24\,e_{42}e_{11} + 24\,e_{42}e_{22} -$$
$$24\,e_{42}e_{33} - 48\,e_{43}e_{32} - 36\,e_{43}^2 - 3\,e_{44}e_{21} + 24\,e_{44}e_{42},$$

$$h_2 = 6\left(e_{21}^2 + 12\,e_{32}e_{31} + 6\,e_{41}e_{11} - 6\,e_{41}e_{22} + 6\,e_{41}e_{33} -\right.$$
$$\left.12\,e_{42}e_{21} + 72\,e_{42}^2 + 12\,e_{43}e_{31} - 6\,e_{44}e_{41}\right),$$

$$h_3 = -2\left(e_{31}^2 - 2\,e_{41}e_{21} + 24\,e_{42}e_{41}\right), \qquad h_4 = 36\,e_{41}^2.$$

It commutes with two third order elements. Under the representation one of them becomes zero and another turns to a differential operator that commutes with P_3.