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Plenary Speakers Abstracts

SDYM equations on the self-dual background. Dressing scheme and the hierarchy.

Leonid Bogdanov
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We introduce the technique combining the features of integration schemes for SDYM equations and multidimensional dispersionless integrable equations to get SDYM equations on the conformally self-dual background. Generating differential form is defined, the dressing scheme is developed. Some special cases and reductions are considered.

Analytic formulae for charge two monopole fields

Victor Enolski
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The ADHMN-construction (Atiyah-Drinfeld-Hitchin-Manin-Nahm) of the Higgs and gauge fields for a nonabelian monopole leads to solving of a Weyl equation, that is a linear ODE with "potentials" given by the so called Nahm data. Even in the case of charge two, where the Nahm data are expressible in terms of elliptic functions, the analytic expressions for the monopole fields are still unknown. We overcome the problem using as we call it *lesser known Nahm Ansatz* comparatively to the well known Nahm Ansatz which reduces ADHM instanton construction to static, ADHMN monopole case. We report complete analytic description in $\mathbb{R}^3$ of charge two monopole fields in terms of four solutions of a quartic (Atiyah-Ward constraint) and four transcendents, given as incomplete second kind elliptic integrals depending in these solutions. We also present analytic expression for the energy density and its visualisation in picture form.

Parabolic equation of normal type connected with 3D Helmholtz system

Andrey Fursikov
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The talk will be devoted to the normal parabolic equation (NPE) connected with $3D$ Helmholtz system whose nonlinear term $B(v)$ is orthogonal projection of nonlinear term for Helmholtz system on the ray generated by vector v. To study NPE of such kind is interesting by the following reasons:

- Its study open the way to construct the method of nonlocal stabilization by feedback control for $3D$ Helmholtz as well as for $3D$ Navier-Stokes equations.
- Its study can help to understand better difficulties that one should overcome to solve Millennium problem on non-local existence of smooth solution for $3D$ Navier-Stokes equations.

The structure of dynamical flow corresponding to this NPE will be described. Besides, the non local stabilization problem for NPE by starting control supported on arbitrary fixed sub-domain will be formulated. The main steps of solution to this problem will be discussed and connection of this problem with non-local stabilization problem for $3D$ Helmholtz system will be explained.

Quantum groups and related algebras: new applications

Dmitry Gurevich
Valenciennes University, France

I'll speak about some algebras equipped with an action of a Quantum Group $U_q(sl(m))$, in particular these, coming to quantum differential calculus. Recently, by using some of them we succeeded to develop a new Noncommutative calculus on the enveloping algebras $U(gl(m))$, including the notion of partial derivatives in their generators. In particular, this enabled us to define Noncommutative counterparts of some dynamical models. I plan to exhibit a Noncommutative version of the Dirac monopole.

Symmetric squares of hyperelliptic curves, associated vector fields and automorphic Lie algebras.

Victor Buchstaber and Alexander Mikhailov
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With the universal space $U_N \subset \mathbb{C}^{N+4}$ of symmetric square of hyperelliptic curves of genus $g = \left[\frac{N}{2}\right]$ we associate a graded ring

$$\mathcal{R} = \mathbb{C}[X_1, X_2, Y_1, Y_2, x_1, \ldots, x_N] \big/ J, \quad \deg x_j = \deg X_k = 2,$$
$$\deg Y_k = N, \; k = 1, 2; \; j = 1, \ldots, N,$$

where $J = \langle \pi_1, \pi_2 \rangle$ is the ideal generated by polynomials

$$\pi_j = Y_j^2 - \prod_{k=1}^{N} (X_j - x_k), \; j = 1, 2, \tag{1}$$

and also the invariant subring $\mathcal{R}^G \subset \mathcal{R}$, where $G = S_N \times S_2$ with respect to the group S_N of permutations of the roots $x_1, \ldots, x_N$ and involution $(X_1, Y_1) \longleftrightarrow (X_2, Y_2)$. We have constructed a graded polynomial Lie algebra $\mathfrak{A}(\mathcal{R}^G)$ of derivations of the coordinate ring $\mathcal{R}$, such that $L : \mathcal{R}^G \mapsto \mathcal{R}^G, \forall L \in \mathfrak{A}(\mathcal{R}^G)$. In algebra $\mathfrak{A}(\mathcal{R}^G)$ we have found N operators $L_{2k}, \; k = -1, 0, \ldots, N-2$ which preserve the ideal $J_\Delta \in \mathcal{R}^G$ generated by the discriminant Δ of the hyperelliptic curve and satisfy the Witt commutation relations $[L_{2p}, L_{2q}] = 2(q-p)L_{2p+2q}$. We identify the ring $\mathcal{R}^G$ with a graded ring $\mathcal{R}_U = \mathbb{C}[u_2, u_4, v_N, v_{N+2}, v_{2N}, y_2, y_4, \ldots, y_{2N}]/I$, where for the ideal I we give Gröbner basis.

Moreover, in the localised algebra $\mathfrak{A}(\mathcal{R}_U[u_4^{-1}])$ we have found two additional operators $L^*_{N-4}, \; L^*_{N-2}$ which are commuting and annihilate functions of variables $x_1, \ldots, x_N$. It is remarkable that two commuting vector fields $L^*_{N-4}, \; L^*_{N-2}$ can be found using a universal construction on symmetric squares of plane curves.

Lemma 1 *Let $F(X,Y)$ be a twice differentiable function and derivations D_k be defined as*

$$D_k = \partial_{Y_k}(F(X_k, Y_k))\partial_{X_k} - \partial_{X_k}(F(X_k, Y_k))\partial_{Y_k}, \qquad k = 1, 2.$$

Then the vector fields

$$\mathcal{L}^1 = \frac{D_1 - D_2}{X_1 - X_2}, \qquad \mathcal{L}^2 = \frac{X_2 D_1 - X_1 D_2}{X_1 - X_2}$$

commute, *have function F in the kernel $\mathcal{L}^i(F(X_j, Y_j)) = 0$ and map symmetric $(X_1, Y_1) \leftrightarrow (X_2, Y_2)$ functions into symmetric.*

We have constructed a ring homomorphism

$$\varphi : \mathcal{R}_U \to \mathcal{A}_N = \mathbb{C}[u_2, u_4, v_{N-2}, v_N, y_2, y_4, \dots, y_{2N-4}],$$

such that

$$\varphi(u_k) = u_k, \qquad k = 2, 4,\ \varphi(v_N) = v_N;\ \ \varphi(y_k) = y_k,\ k = 2, \dots, y_{2N-4},$$
$$\varphi(v_{N+2}) = u_4 v_{N-2},\ \varphi(v_{2N}) = u_4 v_{N-2}^2$$

and polynomials $\varphi(y_{2N-2})$, $\varphi(y_{2N})$ are given explicitly in terms of the Gröbner basis of the ideal I.

Homomorphism φ is a monomorphism and it becomes isomorphism after the localisation with respect to the common variable u_4.

For $N \geq 3$ we have constructed a graded polynomial $\mathcal{A}_N$-Lie algebra $\mathfrak{A}_N$ with a structure of a free $\mathcal{A}_N$–module with $N+2$ generators $L_{2k}\, k = -1, 0, \dots, N-2$, and commuting generators L^*_{N-4}, L^*_{N-2}. Lie algebra $\mathfrak{A}_N$ is given by its faithful representation in $\mathfrak{A}(\mathcal{R}_U[u_4^{-1}])$.

One of our main results is that algebra $\mathfrak{A}_5$ is isomorphic to polynomial $\mathcal{A}_5$–Lie algebra of vector fields on the universal space of Jacobians of curves with genus two, which has been recently constructed in the frame of the theory of two–dimensional σ functions.

Representing the group G in the group Aut $(\mathcal{R} \otimes \mathfrak{G})$, where $\mathfrak{G}$ is a simple Lie algebra we obtain a new class of automorphic $\mathcal{R}^G$ Lie algebras with derivations $\mathfrak{A}_N$.

Acknowledgements

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Explicit formula for Virasoro singular vector

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We denote by Vir the Virasoro algebra with generators $\{C, L_i,\ i \in \mathbb{Z}\}$ and $V(h,c)$ stands for a Verma module over Vir. A nontrivial vector $w \in V(h,c)$ is called singular, if $L_i w{=}0$ for all positive i. There is a singular vector w with grading n in a Verma module $V(h,c)$ if and only if there exist two positive integers p and q and a complex number $t \neq 0$ such that $pq = n$ and

$$c = c(t) = 13 + 6t + 6t^{-1}, \quad h = h_{p,q}(t) = \frac{1-p^2}{4}t + \frac{1-pq}{2} + \frac{1-q^2}{4}t^{-1}.$$

For fixed positive integers p and q and a fixed complex number t the Verma module $V(h_{p,q}(t), c(t))$ containes a homogeneous singular vector $w_{p,q}(t)$ of degree pq defined uniquely up to multiplication by some scalar:

$$w_{p,q}(t) = S_{p,q}(t)v = \sum_{i_1+\dots+i_s=pq} P_{p,q}^{i_1,\dots,i_s}(t) L_{-i_1}\dots L_{-i_s} v,$$

We assume that the coefficient $P_{p,q}^{1,\dots,1}(t)$ is equal to one.

In 1988 Benoist and St-Aubin found an explicit expression for the series $S_{1,p}(t)$:

$$S_{1,p}(t) = \sum_{i_1+\ldots+i_s=p} \frac{(p{-}1)!^2 t^{s-p}}{\prod_{l=1}^{s-1}\Big((\sum_{q=1}^{l} i_q)(p{-}\sum_{q=1}^{l} i_q)\Big)} L_{-i_1}\dots L_{-i_s}, \qquad (2)$$

Later Bauer, Di Francesco, Itzykson and Zuber, using the formula (2) as an initial step, proposed an algorithm for finding all singular vectors. But in practice, their algorithm meets with serious computational difficulties and it is still unclear how to get with its help an explicit formula for all $S_{p,q}(t)$. However, the expressions for $S_{2,2}(t)$ and $S_{2,3}(t)$ found by means of it allowed us to guess the general formula for singular vectors of the entire series $S_{2,p}(t)$.

We prove that for a Verma module $V(h,c)$ over the Virasoro algebra with

$$c{=}c(t){=}13{+}6t{+}6t^{-1},\ h{=}h_{2,p}(t){=}{-}\frac{1}{4}(p{-}1{+}t)(t^{-1}(p{+}1){+}3),\ t \in \mathbb{C}, t \neq 0.$$

we have the following formula for $S_{2,p}(t)$

$$S_{2,p}(t) = \sum_{i_1+\ldots+i_s=2p} f_{2,p}(t; i_1, \ldots, i_s) L_{-i_1} \ldots L_{-i_s}, \qquad (3)$$

where the sums are over all partitions of $2p$ by positive numbers without any ordering restrictions and the coefficients $f_{2,p}(t; i_1, \ldots, i_s)$ are defined by the formulas

$$f_{2,p}(t;i_1,\ldots,i_s)=\frac{(2p-1)!^2(2t)^{s-2p}\prod_{r=1}^{2p-1}(p-t-r)\prod_{m=1}^{s}\left((2t-1)(i_m-1)+2p-1-2\sum_{n=1}^{m-1}i_n\right)}{\prod_{k=0}^{2p-1}(2p-1-2k)\prod_{l=1}^{s-1}\left((\sum_{n=1}^{l}i_n)(2p-\sum_{n=1}^{l}i_n)(p-t-\sum_{n=1}^{l}i_n)\right)}. \qquad (4)$$

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Integrable magnetic geodesic flows on 2-torus: new example via quasi-linear system of PDEs

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The only one example has been known of magnetic geodesic flow on the 2-torus which has a polynomial in momenta integral independent of the Hamiltonian. In this example the integral is linear in momenta and corresponds to a one parametric group preserving the Lagrangian function of the magnetic flow. We consider the problem of integrability on one energy level. This problem can be reduced to a remarkable Semi-hamiltonian system of quasi-linear PDEs and to the question of existence of smooth periodic solutions for this system. Our main result states that the pair of Liouville metric with zero magnetic field on the 2-torus can be analytically deformed to a Riemannian metric with small magnetic field so that the magnetic geodesic flow on an energy level is integrable by means of a quadratic in momenta integral. Thus our construction gives a new example of smooth periodic solution to the Semi-hamiltonian quasilinear system of PDEs.

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The talk is based on the joint paper with Misha Bialy (Tel-Aviv) and Sergey Agapov (Novosibirsk).

Geometry of compatible and almost compatible metrics and integrable systems

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We present the theory of compatible and almost compatible metrics (see [1]–[6]), its applications to integrable systems and some new results on geometry of compatible and almost compatible metrics.

Acknowledgements

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References

[1] O.I. Mokhov. Compatible and almost compatible metrics. Russian Mathematical Surveys. 2000. Vol. 55, no. 4, 819–821.
[2] O.I. Mokhov. Compatible and almost compatible pseudo-Riemannian metrics. Functional Analysis and its Applications. 2001. Vol. 35, no. 2, 100–110.
[3] O.I. Mokhov. Compatible flat metrics. Journal of Applied Mathematics. 2002. Vol. 2, no. 7, 337–370.
[4] O.I. Mokhov. Compatible metrics of constant Riemannian curvature: local geometry, nonlinear equations, and integrability. Functional Analysis and its Applications. 2001. Vol. 36, no. 3, 196–204.
[5] O.I. Mokhov. Riemann invariants of semisimple non-locally bi-Hamiltonian systems of hydrodynamic type and compatible metrics. Russian Mathematical Surveys. 2010. Vol. 65, no. 6, 1183–1185.

[6] O.I. Mokhov. Compatible metrics and the diagonalizability of nonlocally bi-Hamiltonian systems of hydrodynamic type. Russian Mathematical Surveys. 2010. Vol. 65, no. 6, 1183–1185.

Autoresonance in a model of terahertz wave generator

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A model of generator of electromagnetic waves is studied which is based on the stack of Josephson junctions. The model is described by a chain of coupled sine-Gordon equations on phases of electromagnetic field under small dissipation and constant current pumping. We establish the conditions of resonant excitation of the field under various parameters of the system. The auto-resonant nature of Josephson radiation is revealed in the dependence of frequency on the pumping level.

Toric topology and geometry

Taras Panov
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Toric topology is a new area of mathematics that emerged at the end of the 1990s on the border of equivariant topology, algebraic and symplectic geometry, combinatorics, and commutative algebra.
The key players in toric topology are moment-angle manifolds, a class of manifolds with torus actions defined in combinatorial terms. Construction of moment-angle manifolds relates to combinatorial geometry and algebraic geometry of toric varieties via the notion of a quasitoric manifold. Discovery of remarkable geometric structures on moment-angle manifolds led to important connections with classical and modern areas of symplectic, Lagrangian, and non-Kaehler complex geometry. A related categorical construction of moment-angle complexes and polyhedral products provides for a universal framework for many fundamental constructions of homotopical topology. The study of polyhedral products is now evolving into a separate subject of homotopy theory. A new perspective on torus actions has also contributed to the development of classical areas of algebraic topology, such as complex cobordism.

After an introductory part describing the construction and the topology of moment-angle complexes, we shall concentrate on several interesting geometric properties of moment-angle manifolds, emphasising complex-analytic, symplectic and Lagrangian aspects.

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Polynomial and Elliptic Algebras, Heisenberg group and Cremona transformations

Vladimir Rubtsov

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We shall discuss some aspects of polynomial Poisson structures on C^n and $\mathbb{P}^n$ with $n = 3, 4, 5$. The quasi-classical limit of the famous elliptic Sklyanin algebra is a particular important example of such structures. We use the Heisenberg group invariancy and describe a unimodularity property of elliptic Poisson algebras. The case of $n = 5$ is of a special interest because of presence of two non-isomorphic families of Sklyanin elliptic algebras (Odesskii-Feigin). Their relation with Cremona transformations in $\mathbb{P}4$ is described.

Integrable dispersionless PDEs: inverse spectral transform and Cauchy problem; longtime behavior of solutions and wave breaking.

Paolo Santini

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We present the main features of the inverse spectral transform (IST) for integrable dispersionless PDEs in multidimensions, stressing the main differences from the classical IST of soliton PDEs. We use such a novel IST to solve the Cauchy problem, construct the longtime behavior of solutions, and investigate analytically wave breaking phenomena, when they take place. This theory, applicable to several distinguished examples, like the dispersionless Kadomtsev - Petviashvili, the $2D$ dispersionless Toda, and the heavenly equations, has been developed in collaboration with S. V. Manakov, and this talk is dedicated to his memory. Some rigorous aspects of the theory, recently understood in collaboration with P. G. Grinevich and D. Wu, will also be discussed.

Differential operators, geodesics and dynamics of localized quantum states on singular spaces

Andrey Shafarevich

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We discuss properties of evolution equations - for example, Schroedinger or wave equations - on singular spaces. These spaces are obtained from graphs by replacing vertices by small dimensional Riemannian manifolds. Behavior of localized solutions appear to be connected with global properties of geodesic flows on manifolds and with popular problems of analytic number theory.

Deformation quantization and vector fields

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Let (M, π) be a Poisson manifold, where π is a Poisson bivector, i.e. a section of the exterior square of the tangent bundle of M, whose Schouten bracket with itself vanishes. In this case one can introduce the Poisson bracket on functions by the formula:

$$\{f, g\} = \pi(df, dg).$$

Due to the well-known Kontsevich's theorem, one can always find a formal deformation of the algebra of smooth functions on M, i.e. an associative $\hbar$-linear product $*$ in $C^\infty(M)[[\hbar]]$, the space of formal power series with coefficients in $C^\infty(M)$, such that up to higher terms we shall have

$$f * g = fg + \frac{1}{2}\hbar\{f, g\} + \ldots$$

Moreover, this product is uniquely defined up to an equivalence relation. The algebra $\mathcal{A}(M) = (C^\infty)[[\hbar]], *)$ is often called the (formal) deformation quantization of M.

One of important questions, related to this result, can be formulated in the following general form: what other algebraic and geometric structures (such as symmetries, integrable systems, complex or Kaehlerian structures, etc.) can be transferred to the deformation quantization of a manifold? In my talk I am going to address one of the simplest variant of this problem: suppose, there is a Lie algebra $\mathfrak{g}$ acting on M by Poisson fields; is it possible to extend this action to the quantization $\mathcal{A}(M)$, i.e. to find a representation of $\mathfrak{g}$ in the Lie algebra of differentiations on $\mathcal{A}(M)$? It turns out, that this is the case, when $\dim \mathfrak{g} = 1$, but in all other cases there are cohomological obstructions, which should vanish, if the answer is positive. In my talk I will explain the main ideas, that lie behind this result.

Polynomial forms for quantum Calogero-Moser Hamiltonians and commutative sub-algebras in universal enveloping algebras

Vladimir Sokolov

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We find transformations that bring the quantum Calogero-Moser Hamiltonians for $n = 2, 3, 4$ to differential operators with polynomial coefficients. These operators are related to special commutative sub-algebras in the universal enveloping algebras of $gl(n + 1)$.

Bäcklund transformations and Abel equations

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Auto Bäcklund transformation of Hamilton-Jacobi equations is a very particular time-independent transformation of variables, which conserves not only the Hamiltonian character of the equations of motion, but also the Hamilton-Jacobi equations. These canonical transformations preserving the algebraic form of Hamiltonians can be applied to:

- discretization and numerical solution of initial equations of motion;
- construction and investigation of integrable multivalued algebraic maps;
- construction of new integrable systems using hetero Bäcklund transformations;
- classification of the Poisson brackets compatible with the canonical bracket.

If integration by quadratures of the given integrable Hamiltonian systems can be reduced to inversion of Abelian sums of holomorphic hyperelliptic integrals, then an existence of auto Bäcklund transformation is equivalent to an existence of the Abel differential equations. It allows us to use all the machinery developed by Euler, Abel, Jacobi, Weierstrass, Richelot et al to construct. auto Bäcklund transformations for classical integrable systems.

We want to discuss an algebraic construction of the auto Bäcklund transformations for the Hamilton-Jacobi equations solvable by inversion of the Abel quadratures on the hyperelliptic and non-hyperelliptic curves. The main difference between Bäcklund transformations related to hyperelliptic and non-hyperelliptic curves is a valency of the corresponding canonical transformation of variables.

Nonlinear dynamics of spatiotemporal optical waves

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Nonlinear propagation of optical pulses in fibers and waveguides has been so far mostly described by means of 1+1D models such as the NLS equation. On the other hand, the limited capacity of optical fiber transmission channels as a backbone of internet data transmissions requires an upgrade to multimode, and multidimensional optical systems. In this talk I will overview recent theoretical and experimental advances on the dynamics of nonlinear optical waves in multidimensional and multimode systems. The first example will be that of pulse propagation in a dispersive and diffractive waveguide, which is described by means of the 2D-NLSE. Based on the correspondence between solutions of the 2D-NLSE and soliton solutions of the KP equation, I will introduce a new class of optical multidimensional solitary waves of hydrodynamic origin.
Next, I will describe recent experiments of spatio-temporal instabilities and wave condensation phenomena in highly multimode optical fibers, whose propagation is described by the Gross-Pitaievskii equation.

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